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INDEX THEORY FOR BOUNDARY VALUE PROBLEMS VIA CONTINUOUS FIELDS OF C*-ALGEBRAS

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Abstract. We prove an index theorem for boundary value problems in Boutet de Monvel's calculus on a compact manifold X with boundary. The basic tool is the tangent semigroupoid $\mathscr{T}^-X$ generalizing the tangent groupoid defined by Connes in the boundaryless case, and an associated continuous field $C_r^*(\mathscr{T}^-X)$ of C^*-algebras over $[0,1]$. Its fiber in $\hbar=0$, $C_r^*(T^-X)$, can be identified with the symbol algebra for Boutet de Monvel's calculus; for $\hbar\neq 0$ the fibers are isomorphic to the algebra $\mathcal{K}$ of compact operators. We therefore obtain a natural map $K_0(C_r^*(T^-X)) = K_0(\mathcal{C}_0(T^*X)) \to K_0(\mathcal{K}) = \mathbb{Z}$. Using deformation theory we show that this is the analytic index map. On the other hand, using ideas from noncommutative geometry, we construct the topological index map and prove that it coincides with the analytic index map.

Introduction

Let X be a smooth compact manifold with boundary ∂X. An operator in Boutet de Monvel's calculus on X is a matrix

$$A = \begin{pmatrix} P^+ + G & K \\ T & S \end{pmatrix} : \begin{matrix} \mathcal{C}^\infty(X,E_1) \\ \oplus \\ \mathcal{C}^\infty(\partial X,F_1) \end{matrix} \to \begin{matrix} \mathcal{C}^\infty(X,E_2) \\ \oplus \\ \mathcal{C}^\infty(\partial X,F_2) \end{matrix} . \tag{0.1}$$

of operators acting on smooth sections of (hermitian) vector bundles E_1 and E_2 over X and F_1 and F_2 over ∂X.

Here P is a pseudodifferential operator on the double $\widetilde{X}$ of X, and P^+ is the so-called truncated operator given by $P^+ = r^+Pe^+$, where $e^+ : \mathcal{C}^\infty(X,E_1) \to L^2(\widetilde{X},E_1)$ denotes extension by zero and r^+ is the operator of restriction of distributions on $\widetilde{X}$ to the open interior X° of X.

For a general pseudodifferential operator P, the truncation P^+ will map $\mathcal{C}^\infty(X,E_1)$ to $\mathcal{C}^\infty(X^\circ,E_2)$, but the result may not be smooth up to the boundary. The operator P is supposed to satisfy the transmission condition to ensure mapping property (0.1).

The entry G is a so-called singular Green operator. Roughly speaking it acts like an operator-valued pseudodifferential operator along the boundary with values in smoothing operators in the normal direction. In the interior, G is regularizing. Singular Green operators come up naturally: If P and Q are pseudodifferential operators on X, then the composition of the associated truncated operators differs from the truncation of the composition by the so-called leftover term

$$L(P,Q) = (PQ)^+ - P^+Q^+, \tag{0.2}$$

which is a singular Green operator.

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The operators T and K are trace and potential (or Poisson) operators, respectively, and S is a pseudodifferential operator on ∂X. We skip details here, since for the purpose of index theory it is sufficient to consider the case where there are no bundles over the boundary and the operator A is of the form

$$A = P^+ + G$$

with both P and G classical. Moreover we can confine ourselves to the case where A is of order and class zero.

In analogy to the classical Lopatinskij-Shapiro condition, the Fredholm property is governed by the invertibility of two symbols. The first, the interior symbol of A, simply is the principal symbol of P. The second, the boundary symbol, is an operator-valued symbol on $S^*\partial X$ which we will explain, below.

The above calculus was introduced by L. Boutet de Monvel in 1971, [2]. He showed that the operators of the form (0.1) indeed form an algebra under composition, assuming the vector bundles match. Moreover, this calculus contains the parametrices to elliptic elements and even their inverses whenever they exist.

He also proved an index theorem: To every elliptic operator A one can associate a class $[A]$ in $K_c(T^*X^\circ)$, the compactly supported K-theory of the cotangent bundle over the interior X° of X. The index of A then is given by the topological index of $[A]$. It is far from obvious how to assign $[A]$ to A. Boutet de Monvel gave a brilliant construction, combining pseudodifferential analysis and classical topological K-theory in an ingenious way. Still, the proof is hard for readers not familiar with the details of the calculus, and efforts have been made to make it accessible to a wider audience.

Following work by Melo, Nest, and Schrohe [8], it has been shown in [9] by Melo, Schick and Schrohe that the mapping $A \mapsto [A]$ can be obtained more easily with the help of C^*-algebra K-theory, which had not yet been developed in 1971. This proof relies on only a rudimentary knowledge of the calculus and the ideal structure of the algebra of operators of order and class zero, but it lacks the geometric intuition of Boutet de Monvel's initial idea.

In this paper we will make the link to geometry. We will show how the index of a Fredholm operator of order and class 0 can be determined from its two symbols with the help of deformation theory and a continuous field of C^*-algebras over $[0,1]$.

This field, $C^*_r(\mathscr{T}^-X)$, is the reduced C^*-algebra of the tangent semigroupoid $\mathscr{T}^-X$ associated to X; it was introduced in [1]. The construction of $\mathscr{T}^-X$ is similar to Connes' construction of the tangent groupoid for a closed manifold, cf. [3, Section II.5]. In the case at hand, the 'half-tangent space' T^-X is glued to $X \times X \times]0,1]$. Here, fixing a connection, $T^\pm X$ consists of all tangent vectors (x,v) such that $\exp_x(\pm tv) \in X$ for small $t \geq 0$, and $X \times X \times]0,1]$ is endowed with the pair groupoid structure; the gluing is performed with the help of the exponential map: $(x,v,\hbar) \mapsto (x, \exp_x(-\hbar v), \hbar)$.

We showed in [1] that – just as in the boundaryless case – the fibers of $C^*_r(\mathscr{T}^-X)$ are isomorphic to the algebra $\mathcal{K}$ of compact operators for $\hbar > 0$ so that their K-theory is given by $\mathbb{Z}$. For $\hbar = 0$, the situation is different. The 'symbol algebra' $C^*_r(\mathscr{T}^-X)(0) = C^*_r(T^-X)$ is generated by two representations of $\mathcal{C}^\infty_c(T^-X)$. The first is the representation on $L^2(X^\circ)$ via convolution in the fibers. The second takes into account the boundary: We associate to $f \in \mathcal{C}^\infty_c(T^-X)$ the operator $\pi_0^\partial(f)$ on $L^2(T^+X|_{\partial X})$ given by half-convolution:

$$\pi_0^\partial(f)\xi(x,v) = \int_{T^+X} f(x, v-w)\xi(w)\, dw.$$

The K-theory of $C_r^*(T^-X)$ turned out to be given by $K_0(\mathcal{C}_0(T^*X)) = K_c(T^*X^\circ)$.

In order to make use of this, we first compose the operator with an order reducing operator of positive order $m > \dim X$. This gives us an operator A of order m and class 0 in Boutet de Monvel's calculus (in fact, we might have started with A this order and class).

We denote by p^m its principal pseudodifferential symbol, a homogeneous function on the nonzero vectors in the cotangent bundle, and by c^m its homogeneous principal boundary symbol, a homogeneous operator-valued function defined outside the zero section in the cotangent bundle over the boundary.

We then set out to compute the index of A. To this end we first smooth out both symbols near the zero section in the corresponding cotangent bundle, obtaining a smooth function p on T^*X and a smooth boundary symbol operator c on $T^*\partial X$.

Following an idea of Elliott-Natsume-Nest [4] we then consider a semiclassical deformation $A_\hbar,\ 0 < \hbar \le 1$ of A with $A = A_1$ and study the associated graph projection

$$\mathcal{G}(A_\hbar) = \begin{pmatrix} (1 + A_\hbar^* A_\hbar)^{-1} & (1 + A_\hbar^* A_\hbar)^{-1} A_\hbar^* \\ A_\hbar (1 + A_\hbar^* A_\hbar)^{-1} & A_\hbar (1 + A_\hbar^* A_\hbar)^{-1} A_\hbar^* \end{pmatrix}. \tag{0.3}$$

The positivity of the order of A is crucial here; it ensures that all entries of this matrix are compact operators, except for the one in the lower right corner which differs from a compact operator by the identity. The graph projection of $A_\hbar$ therefore is a projection in $\mathcal{K}^\sim$, the unitization of the compact operators. It is closely related to the index of A. In fact, denoting by e the projection $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ (we will use the same notation for this projection in various algebras), we will show in Theorem 4.15 that

$$[\mathcal{G}(A)] - [e] = [\pi_{\ker A}] - [\pi_{\operatorname{coker} A}] \tag{0.4}$$

is the difference of the classes associated to the projections onto the kernel and the cokernel, respectively, thus the index of A.

Using formula (0.3) we can also define the graph projections for p and c.

The crucial step is to show that

$$\hbar \mapsto \begin{cases} \mathcal{G}(A_\hbar), & 0 < \hbar \le 1 \\ \mathcal{G}(p) \oplus \mathcal{G}(c), & \hbar = 0 \end{cases}$$

defines a continuous section of the unitization of $C_r^*(\mathscr{T}^-X)$, cf. Proposition 6.24, Theorem 6.25.

A technical problem arises from the fact that Boutet de Monvel's calculus does not in general contain the adjoints of operators of positive order. The analysis of the graph projection therefore takes us out of the calculus. We overcome this difficulty by working with operator-valued symbol classes. Boutet de Monvel's calculus fits well into this concept, cf. Schrohe and Schulze [14], Schrohe [13]; moreover, it also allows to treat the adjoints.

The continuity of the section gives us a natural map associating to an elliptic operator A a class in $K_0(C_r^*(T^-X)) = K_0(\mathcal{C}_0(T^*X^\circ)) = K_c(T^*X^\circ)$ by evaluating the section in $\hbar = 0$, more precisely by taking

$$([\mathcal{G}(p) \oplus \mathcal{G}(c)]) - ([e \oplus e]) \,.$$

In addition, evaluation in $\hbar = 0$ and in $\hbar = 1$ defines a map in K-theory

$$\operatorname{ind}_a : K_0(\mathcal{C}_0(T^*X^\circ)) \to K_0(\mathcal{K});$$

it associates to $([\mathcal{G}(p) \oplus \mathcal{G}(c)]) - ([e \oplus e])$ the class $[\mathcal{G}(A)] - [e]$, thus the index of A.

In this way we obtain the analytic index map. In a second step we then construct the topological index in order to obtain an index formula in cohomological terms.

A cohomological index formula had been established before by Fedosov [5, Chapter 2, Theorem 2.4]: He showed that

$$\operatorname{ind} A = \int_{T^*X} \operatorname{ch}([p]) \operatorname{Td}(X) + \int_{T^*\partial X} \operatorname{ch}'([c]) \operatorname{Td}(X),$$

with the Chern character of the K-theory class of p and a variant ch' of the Chern character of the K-class of c. As usual, Td denotes the Todd class.

In this article, we conclude the discussion in the spirit of noncommutative geometry. We extend the fundamental class

$$\int_{T^*X^\circ} : H^*_c(T^*X^\circ) = HP^*(\mathcal{C}^\infty_c(T^*X^\circ) \to \mathbb{C}$$

to a fundamental class

$$F : HP^*(\mathcal{C}^\infty_{tc}(T^-X)) \to \mathbb{C}.$$

Here

$$\mathcal{C}^\infty_{tc}(T^-X) = \mathcal{C}^\infty_c(TX) \oplus \mathcal{C}^\infty_c(T\partial X \times \mathbb{R}_+ \times \mathbb{R}_+)$$

is the 'smooth' algebra associated to the symbol algebra $C^*_r(T^-X)$ which was introduced in [1, Definition 2.16].

The construction leaves us a certain degree of freedom. In fact, we obtain extensions F_ω for every choice ω of a closed form on T^*X of even degree, which is the pull-back of a closed form on X.

Using the equality of the analytical and the topological index in the boundaryless case, established by Connes, we then obtain the index formula

$$\operatorname{ind} A = F_{\operatorname{Td}(X)}(\operatorname{ch}([\mathcal{G}(p) \oplus \mathcal{G}(c)] - [e \oplus e]))$$

with the Chern-Connes character ch.

1. Operators in Boutet de Monvel's Calculus

1.1. Manifolds with boundary. In the sequel let X be a compact manifold with boundary and $\widetilde{X}$ its double. We use the standard Sobolev space $H^s(\widetilde{X})$ and the associated spaces

$$H^s(X) = \{u|_{X^\circ} : u \in H^s(\widetilde{X})\}, \quad H^s_0(X) = \{u \in H^s(\widetilde{X}) : \operatorname{supp} u \subseteq X\}.$$

The space $\mathcal{C}^\infty(X)$ is dense in $H^s(X)$ for all s, while $\mathcal{C}^\infty_c(X^\circ)$ is dense in $H^s_0(X)$ for all s. The L^2-inner product allows us to identify $H^s(X)$ with the dual of $H^{-s}_0(X)$.

In general, the spaces $H^s(X)$ and $H^s_0(X)$ are quite different. For $-1/2 < s < 1/2$, however, they can be identified. In particular, we have for $s = 0$ the natural identification of $L^2(X)$ with the subset of all functions in $L^2(\widetilde{X})$ which vanish on $\widetilde{X} \setminus X$.

1.2. Operators, symbols, ellipticity. Detailed descriptions of Boutet de Monvel's calculus were given by Grubb [6] and Rempel and Schulze [10]. In order to overcome technical difficulties we will also rely on the representation of these operators by operator-valued symbols as presented in [13].

In order to keep the exposition short, we will focus on the algebra formed by the elements in the upper left corner:

$$\mathcal{A} = \{A : \mathcal{C}^\infty(X, E_1) \to \mathcal{C}^\infty(X, E_2) : A = P^+ + G\},$$

where P is a pseudodifferential operator with the transmission property and G a singular Green operator (sGo). We assume all operators to be classical.

The operator A is said to have order μ and class $d \in \mathbb{N}_0$, if P is of order μ and G is of order μ and class d. We speak of smoothing or regularizing operators, if the order is $-\infty$.

A sGo G of order m and class d can be written

$$G = \sum_{j=0}^{d} G_j \partial_\nu^j$$

where G_j, $j = 0, \ldots d$, are sGo's of order $\mu - j$ and class 0 and ∂_ν is a differential operator which coincides with the normal derivative in a neighborhood of the boundary and vanishes farther away from the boundary. This representation differs from the standard one in that it avoids the trace operators. The equivalence becomes clear from 2.4, below.

Let $\varphi \in \mathcal{C}_c^\infty(X^\circ)$, and denote by M_φ multiplication by φ. Then GM_φ is regularizing of class 0 and $M_\varphi G$ is regularizing of class d.

We associate two symbols to A. The first is the pseudodifferential principal symbol, $\sigma_\psi^\mu(A)$. It is defined as the principal symbol of P, restricted to $T^*X \setminus 0$:

$$\sigma_\psi^\mu(A) = \sigma^\mu(P)|_{T^*X\setminus 0}\,.$$

This makes sense as G is smoothing in X° and therefore cannot contribute to the symbol.

The second is the principal boundary symbol, $\sigma_\partial^\mu(A)$, defined on $T^*\partial X \setminus 0$. In local coordinates near ∂X, it is given by

$$\sigma_\partial^\mu(A)(x', \xi') = p^\mu(x', 0, \xi', D_n)^+ + g^\mu(x', \xi', D_n) : \mathcal{S}(\overline{\mathbb{R}}_+, \tilde{E}_1) \to \mathcal{S}(\overline{\mathbb{R}}_+, \tilde{E}_2).$$

Here p^μ and g^μ are the homogeneous principal symbols of P and G, respectively. For fixed x', ξ', the operator $p^\mu(x', 0, \xi', D_n)$ is the Fourier multiplier with symbol $p^\mu(x', 0, \xi', \xi_n)$, while $g^\mu(x', \xi', D_n)$ is an integral operator with smooth integral kernel. By $\tilde{E}_j$ we have denoted the fiber in (x', ξ') of the pullback of E_j to T^*X.

The principal boundary symbol is homogeneous on $T^*\partial X \setminus 0$ in a sense we shall explain later, see (2.7), so that it can be viewed as a function on $S^*\partial X$.

An operator A of order μ and class $d \le \max\{\mu, 0\}$ is said to be elliptic, if

(i) $\sigma_\psi^\mu(A)(x, \xi) : \pi^* E_1 \to \pi^* E_2$ is invertible for all $(x, \xi) \in T^*X \setminus 0$, and

(ii) $\sigma_\partial^\mu(A)(x', \xi') : H^\mu(\mathbb{R}_+, \widetilde{E}_1) \to L^2(\mathbb{R}_+, \widetilde{E}_2)$ is invertible for all $(x', \xi') \in T^*\partial X \setminus 0$.

Here $\pi : T^*X \to X$ is the base point projection.

Apart from these symbols we have, of course, in any coordinate neighborhood, the full symbols of P and G in the corresponding classes.

1.3. Theorem. *Let A be an operator of order μ and class d in Boutet de Monvel's calculus. Then A induces a bounded linear map*

$$A : H^s(X, E_1) \to H^{s-\mu}(X, E_2)$$

for each $s > d - 1/2$.

In general we cannot extend A to $H^s(X, E_1)$ for $s \le d - 1/2$. The reason is that neither extension by zero makes sense on these spaces nor do integral operators with smooth (up to the boundary) integral kernels act continuously on them.

1.4. Theorem. *Let $A_1 : \mathcal{C}^\infty(X, E_1) \to \mathcal{C}^\infty(X, E_2)$ and $A_2 = \mathcal{C}^\infty(X, E_2) \to \mathcal{C}^\infty(X, E_3)$ be operators of orders μ_1 and μ_2 and classes d_1 and d_2, respectively, in Boutet de Monvel's calculus on X. The composition A_2A_1 is an operator of order $\mu_1+\mu_2$ and class $\max\{d_1, \mu_1+d_2\}$. Its principal symbols are given by*

$$\begin{aligned}\sigma_\psi^{\mu_1+\mu_2}(A_2A_1) &= \sigma_\psi^{\mu_2}(A_2)\,\sigma_\psi^{\mu_1}(A_1);\\ \sigma_\partial^{\mu_1+\mu_2}(A_2A_1) &= \sigma_\partial^{\mu_2}(A_2)\,\sigma_\partial^{\mu_1}(A_1).\end{aligned}$$

1.5. Theorem. *Let A be an operator of order μ and class $d \le \max\{\mu, 0\}$ in Boutet de Monvel's calculus and $s > d - 1/2$. Then*

$$A : H^s(X, E_1) \to H^{s-\mu}(X, E_2)$$

is Fredholm if and only if A is elliptic. In this case we find an operator B of order $-\mu$ and class $d' \le \max\{-\mu, 0\}$ such that

$$AB = I + R_1 \quad \text{and} \quad BA = I + R_2$$

with regularizing operators R_1 and R_2 of class d' and d, respectively. For the symbols we have

$$\sigma_\psi^{-\mu}(B) = \sigma_\psi^{\mu}(A)^{-1} \quad \text{and} \quad \sigma_\partial^{-\mu}(B) = \sigma_\partial^{\mu}(A)^{-1}.$$

1.6. Adjoints. Boutet de Monvel's calculus is not closed under taking adjoints. For the sake of completeness let us introduce a few basic concepts. To an operator A of order μ and class $d \le \max\{0, \mu\}$ we can associate a minimal adjoint $A^*_{\min}$ defined on $\mathcal{C}_c^\infty(X^\circ)$ (we omit the bundles from the notation) and taking values in $\mathcal{C}^\infty(X)'$ by the relation

$$\langle Au, v\rangle = \langle u, A^*_{\min}v\rangle, \quad u \in \mathcal{C}^\infty(X), v \in \mathcal{C}_c^\infty(X^\circ).$$

For $s > d - 1/2$, the adjoint A^* of the bounded operator

$$A : H^s(X) \to H^{s-\mu}(X)$$

is then given by extending $A^*_{\min}$ by continuity to an operator $A^* : H_0^{\mu-s}(X) \to H_0^{-s}(X)$. If the class is zero, we can explicitly determine the adjoint: We write $A = P^+ + G$ with a pseudodifferential operator P on $\widetilde{X}$ and a singular Green operator G of order μ and class 0. Next we recall from [6, Lemma 1.3.1] that a pseudodifferential operator P of order $\mu \in \mathbb{Z}$ with the transmission property can be written

$$P = S + Q \tag{1.1}$$

where S is a differential operator of order $\le \mu$ and Q is a pseudodifferential operator of order μ which maps $e^+\mathcal{C}^\infty(X)$, the extensions (by zero) of smooth functions on X to $\widetilde{X}$, to $L^2(X)$ and satisfies – with the formal adjoint Q_f^* –

$$\langle Q^+u, v\rangle = \langle u, Q_f^{*,+}v\rangle, \quad u, v \in \mathcal{C}_c^\infty(X). \tag{1.2}$$

Note that for $\mu \le 0$, equality (1.2) will hold for P. Still it it useful to know that we can choose S in such a way that the local symbol q of Q is of order (at most) -1 with respect to ξ_n; we say that q is of normal order -1.

For $u, v \in \mathcal{C}^\infty(X)$ it is known [6, Section 1.6] that

$$\langle S^+u, v\rangle_{L^2(X)} = \langle u, S_f^{*,+}v\rangle_{L^2(X)} + \langle \mathfrak{A}\rho^+u, \rho^+v\rangle_{L^2(\partial X)}$$

with the formal adjoint S_f^* of S, the Green matrix $\mathfrak{A}$, and the vectors ρ^+u and ρ^+v of boundary values for u and v. When v lies in $\mathcal{C}_c^\infty(X^\circ)$, the boundary terms on the right hand side vanish, so that

$$\langle S^+u, v\rangle_{L^2(X)} = \langle u, S_f^{*,+}v\rangle_{L^2(X)}, \quad u \in \mathcal{C}^\infty(X), v \in \mathcal{C}_c^\infty(X^\circ).$$

On $\mathcal{C}_c^\infty(X^\circ)$, the operation e^+ of extending by zero is trivial; moreover S_f^* is a differential operator, so that $S_f^*v \in \mathcal{C}_c^\infty(X^\circ)$. There are no singular terms arising at the boundary, and $S_f^{*,+}v = S_f^*v$ as a functional on $\mathcal{C}^\infty(X)$. For Q the corresponding identity (1.2) is valid by construction. Again, $Q_f^{*,+}v = Q_f^*v$ as a functional on $\mathcal{C}^\infty(X)$. Hence

$$\langle P^+u, v\rangle_{L^2(X)} = \langle u, P_f^*v\rangle_{L^2(X)}, \quad u \in \mathcal{C}^\infty(X), v \in \mathcal{C}_c^\infty(X^\circ). \tag{1.3}$$

As the singular Green part G is assumed to be of class zero, it also has a formal adjoint G_f^*, cf. [6, (1.2.47)], and

$$\langle Gu, v\rangle_{L^2(X)} = \langle u, G_f^*v\rangle_{L^2(X)}. \tag{1.4}$$

Hence, as a functional on $\mathcal{C}^\infty(X)$,

$$A_{\min}^*v = P_f^*v + G_f^*v, \quad v \in \mathcal{C}_c^\infty(X^\circ).$$

If $A = P^+ + G$ is of order $\mu \le 0$ and class 0, then $A : L^2(X) \to L^2(X)$ is continuous, and $\mathcal{C}_c^\infty(X^\circ)$ is dense in $L^2(X)$. We conclude from (1.3) and (1.4) that the L^2-adjoint of A is

$$A^* = P_f^{*,+} + G_f^*$$

with the formal adjoints P_f^* and G_f^* of P and G, extended to L^2. Thus A^* again is an operator in Boutet de Monvel's calculus. Its principal symbols are given by

$$\begin{aligned} \sigma_\psi^{\mu_1}(A^*) &= \sigma_\psi^{\mu_1}(A)^*; \\ \sigma_\partial^{\mu_1}(A^*) &= \sigma_\partial^{\mu_1}(A)^*. \end{aligned}$$

1.7. Corollary. *Suppose $A : H^\mu(X, E_1) \to L^2(X, E_2)$ is invertible of order $\mu \ge 0$ and class $d \le \mu$. Then A^{-1} is an operator of order $-\mu$ and class 0, cf.* [12, Theorem 4.5]. *It has an adjoint $(A^{-1})^*$ in Boutet de Monvel's calculus, which extends to a bounded operator from $H_0^{-\mu}(X, E_2)$ to $L^2(X, E_1)$, considering e^+ as a trivial operation on $H_0^{-\mu}$. On the other hand, the minimal adjoint $A_{\min}^*$ of A extends to an invertible operator $A^* : L^2(X, E_2) \to H_0^{-\mu}(X, E_1)$. It is easily checked that $(A^{-1})^* = (A^*)^{-1}$.*

1.8. Order reducing operators. There exists a family Λ_-^m, $m \in \mathbb{Z}$, of classical scalar pseudodifferential operators on X satisfying the transmission condition with the following properties:

(i) $(\Lambda_-^m)^+ : H^s(X) \to H^{s-m}(X)$ is an isomorphism for all $s > -1/2$.

(ii) In fact, the result of the application of $r^+\Lambda_-^m$ to u in $H^s(\widetilde{X})$ only depends on the restriction of u to X°. The map in (i) extends to an isomorphism $r^+\Lambda_-^m e_s : H^s(X) \to H^{s-m}(X)$ for an arbitrary choice of an extension operator $e_s : H^s(X) \to H^s(\widetilde{X})$.

(iii) $(\Lambda_-^m)^+(\Lambda_-^\mu)^+ = (\Lambda_-^{m+\mu})^+$, $m, \mu \in \mathbb{Z}$.

(iv) The (extension of the) formal pseudodifferential adjoint defines an isomorphism $\Lambda_+^m = ((\Lambda_-^m)^+)^* : H_0^s(X) \to H_0^{s-m}(X)$.

(v) The operators Λ_-^m, Λ_+^m can be extended to operators with the same properties but acting in a vector bundle E.

2. Operator-valued Symbols

It will be helpful to consider the operators in Boutet de Monvel's calculus as operator-valued pseudodifferential operators. We recall the basic concepts from [13].

We first fix a function $\mathbb{R}^n \ni \xi \mapsto [\xi] \in \mathbb{R}_{\geq 0}$ which is positive for $\xi \neq 0$ and coincides with $|\xi|$ for $|\xi| \geq 1$.

2.1. Group actions. A strongly continuous group action on a Banach space E is a family $\kappa = \{\kappa_\lambda : \lambda \in \mathbb{R}_+\}$ of isomorphisms in $\mathcal{L}(E)$ such that $\kappa_\lambda \kappa_\mu = \kappa_{\lambda\mu}$ and the mapping $\lambda \mapsto \kappa_\lambda e$ is continuous for every $e \in E$. Note that there is an $M > 0$ such that

$$\|\kappa_\lambda\| \leq \left(\max\{\lambda, \lambda^{-1}\}\right)^M . \tag{2.1}$$

For the usual Sobolev spaces on $\mathbb{R}$ and $\mathbb{R}_+$ we shall use the group action defined on functions u by

$$(\kappa_\lambda u)(x) = \lambda^{1/2} u(\lambda x). \tag{2.2}$$

It will be useful to consider also weighted Sobolev spaces: For $s = (s_1, s_2) \in \mathbb{R}^2$ we define

$$H^s(\mathbb{R}) = H^{(s_1,s_2)}(\mathbb{R}) = \{[x]^{-s_2} u : u \in H^{s_1}(\mathbb{R})\}$$

with the usual unweighted space on the right hand side. Similarly we define $H^s(\mathbb{R}_+)$. We then have $\mathcal{S}(\mathbb{R}_+) = \operatorname{projlim}_{s_1,s_2\to\infty} H^{s_1,s_2}(\mathbb{R}_+)$ and $\mathcal{S}'(\mathbb{R}_+) = \operatorname{indlim}_{s_1,s_2\to\infty} H^{-s_1,-s_2}(\mathbb{R}_+)$. On $E = \mathbb{C}^l, l \in \mathbb{N}$, we use the trivial group action $\kappa_\lambda \equiv \mathrm{id}$. Sums of spaces of the above kind will be endowed with the sum of the group actions.

2.2. Operator-valued symbols and amplitudes. Let E, F be Banach spaces with strongly continuous group actions κ and $\tilde{\kappa}$, respectively. Let $a \in \mathcal{C}^\infty(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q, \mathcal{L}(E,F))$ and $\mu \in \mathbb{R}$. We shall write $a \in S^\mu(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; E, F)$ and call a an amplitude of order μ provided that, for all multi-indices α, β, γ, there is a constant $C = C(\alpha,\beta,\gamma)$ with

$$\|\tilde{\kappa}_{[\eta]^{-1}} D_\eta^\alpha D_y^\beta D_{\tilde{y}}^\gamma a(y, \tilde{y}, \eta) \kappa_{[\eta]}\|_{\mathcal{L}(E,F)} \leq C\, [\eta]^{\mu - |\alpha|}.$$

If a is independent of y or $\tilde{y}$ we shall write $a \in S^\mu(\mathbb{R}^q \times \mathbb{R}^q; E, F)$.

For $E = F = \mathbb{C}$ we recover the usual pseudodifferential symbol classes.

The concept extends to the cases where E is an inductive or F a projective limit.

In order to avoid lengthy formulas we shall abbreviate this by saying that a is a symbol of order μ with values in $\mathcal{L}(E,F)$.

2.3. Example: Potential, trace and singular Green boundary symbol operators. The boundary symbol operators associated to potential, trace or singular Green symbols in the usual presentation of Boutet de Monvel's calculus have simple descriptions in the framework of operator-valued symbols.

(a) The elements in $S^\mu(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathbb{C}, \mathcal{S}(\mathbb{R}_+))$ are precisely the boundary symbol operators associated with potential symbols of order [1] μ on $\overline{\mathbb{R}}^n_+$.

(b) The elements of $S^\mu(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{S}'(\mathbb{R}_+), \mathbb{C})$ are precisely the boundary symbol operators associated with trace symbols of order μ and class 0. Those of the form

$$\sum_{j=0}^{d} t_j(x', \xi') D_n^j$$

[1]In fact, the notion of order differs slightly in [6], [13] and [14]; this will, however, not play a role in the sequel.

with t_j in $S^{\mu-j}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{S}'(\mathbb{R}_+),\mathbb{C})$ and the derivative D_n on $\mathbb{R}$ are the boundary symbol operators associated with trace symbols of order μ and class d on $\overline{\mathbb{R}}^n_+$.

(c) The elements of $S^\mu(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{S}'(\mathbb{R}_+),\mathcal{S}(\mathbb{R}_+))$ are precisely the boundary symbol operators associated with singular Green symbols of order μ and class 0 on $\overline{\mathbb{R}}^n_+$. Those of the form

$$\sum_{j=0}^{d} g_j(x',\xi')D_n^j$$

with g_j in $S^{\mu-j}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{S}'(\mathbb{R}_+),\mathcal{S}(\mathbb{R}_+))$ are the singular Green boundary symbol operators of order μ and class d.

We therefore speak of these operator-valued symbols as potential, trace and singular Green boundary symbol operators or, for short, symbols, of the corresponding orders and classes.

2.4. Example: Trace operators. Let γ_j be defined on $\mathcal{S}(\mathbb{R}_+)$ by $\gamma_j u=\lim_{t\to 0^+}D_n^j u(t)$. It extends to an element of $\mathcal{L}(H^\sigma(\mathbb{R}_+),\mathbb{C})$ for $\sigma=(\sigma_1,\sigma_2)$, $\sigma_1>j+1/2$, by the trace theorem for Sobolev spaces.

Viewed as an operator-valued symbol independent of the variables y and η, γ_j then is a symbol of order $j+1/2$ with values in $\mathcal{L}(H^\sigma(\mathbb{R}_+),\mathbb{C})$: Recalling that the group action on the Sobolev space is given by (2.2) while on $\mathbb{C}$ it is given by the identity, we only have to check that

$$\|\gamma_j\kappa_{[\eta]}\|_{\mathcal{L}(H^\sigma(\mathbb{R}_+),\mathbb{C})}=O([\eta]^{j+1/2}).$$

This is immediate, since

$$\partial_t^j\left([\eta]^{1/2}u([\eta]t)\right)=[\eta]^{j+1/2}(\partial_t^j u)([\eta]t).$$

Let us next show that γ_j is a trace symbol of order $j+1/2$ and class $j+1$ in the sense of 2.3. It clearly suffices to do this for $j=0$.

Choose $\varphi\in\mathcal{S}(\mathbb{R}_+)$ with $\varphi(0)=1$. The identity

$$u(0)=-\int_0^\infty[\xi']\varphi'([\xi']s)u(s)\,ds-\int_0^\infty\varphi([\xi']s)\partial_s u(s)\,ds,\ u\in\mathcal{S}(\mathbb{R}_+),\tag{2.3}$$

shows that

$$\gamma_0=t_0+it_1D_n,$$

where t_0 and t_1 are the operator-valued symbols of order $1/2$ and $-1/2$, respectively, with values in $\mathcal{L}(\mathcal{S}'(\mathbb{R}_+),\mathbb{C})$, given by

$$t_0u=-\int_0^\infty[\xi']\varphi([\xi']s)u(s)\,ds,\quad t_1u=-\int_0^\infty\varphi([\xi']s)u(s)\,ds.$$

Hence γ_0 is of class 1.

2.5. Definition. For $a\in S^\mu(\mathbb{R}^q\times\mathbb{R}^q\times\mathbb{R}^q;E,F)$, the pseudodifferential operator

$$\operatorname{op}a:\mathcal{S}(\mathbb{R}^q,E)\to\mathcal{S}(\mathbb{R}^q,F)$$

is defined by

$$(\operatorname{op}a)\,u(y)=\iint e^{i(y-\tilde y)\eta}a(y,\tilde y,\eta)u(\tilde y)\,d\tilde y\,đ\eta;\qquad y\in\mathbb{R}^q.$$

Here $đ\eta=(2\pi)^{-q}d\eta$. If a is independent of $\tilde y$, this reduces to

$$(\operatorname{op}a)\,u(y)=\int e^{iy\eta}a(y,\eta)\hat u(\eta)\,đ\eta;\tag{2.4}$$

in this case we call a a *left* symbol for $\operatorname{op} a$. If a is independent of y, then

$$(\operatorname{op} a)\, u(y) = \iint e^{i(y-\tilde y)\eta} a(\tilde y, \eta) u(\tilde y)\, d\tilde y\, đ\eta, \tag{2.5}$$

and a is called a *right* symbol.

2.6. Example: Action in the normal direction. Let $p \in S^\mu(\mathbb{R}^n \times \mathbb{R}^n)$. For fixed (x', ξ'), the function $p(x', \cdot, \xi', \cdot)$ is an element of $S^\mu(\mathbb{R} \times \mathbb{R})$. For $\sigma \in \mathbb{R}^2$, $p(x', \cdot, \xi', \cdot)$ induces a bounded linear operator

$$p(x', x_n, \xi', D_n) = (\operatorname{op}_{x_n} p)(x', \xi') : H^\sigma(\mathbb{R}) \to H^{(\sigma_1-\mu, \sigma_2)}(\mathbb{R});$$

by

$$(\operatorname{op}_{x_n} p)(x', \xi') u(x_n) = \int e^{ix_n\xi_n} p(x', x_n, \xi', \xi_n) \hat u(\xi_n)\, đ\xi_n,$$

see [11, Theorem 1.7] for the boundedness on weighted spaces. We then have

$$\kappa_{[\xi']^{-1}} \left(\operatorname{op}_{x_n} p\right) \kappa_{[\xi']} = \operatorname{op}_{x_n} p\left(x', x_n/[\xi'], \xi', [\xi']\xi_n\right) : \tag{2.6}$$

In fact, for $u \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned} &\kappa_{[\xi']^{-1}} \left(\operatorname{op}_{x_n} p\right) (\kappa_{[\xi']} u)(x_n) \\ &= \int e^{ix_n\xi_n/[\xi']} [\xi']^{-1} p(x', x_n/[\xi'], \xi', \xi_n) \hat u(\xi_n/[\xi'])đ\xi_n; \end{aligned}$$

and the substitution $\eta_n = \xi_n/[\xi']$ yields the assertion. The theorem, below, shows that $\operatorname{op}_{x_n} p$ is an operator-valued symbol in the sense of 2.2:

2.7. Proposition. *In the above situation we have*

$$\operatorname{op}_{x_n} p \in S^\mu(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; H^\sigma(\mathbb{R}), H^{(\sigma_1-\mu, \sigma_2)}(\mathbb{R})).$$

Proof. Given multi-indices α, β, we have to estimate

$$\begin{aligned} &\sup_{x', \xi'} \| [\xi']^{|\alpha|} \kappa_{[\xi']^{-1}} (\operatorname{op}_{x_n} (D^\alpha_{\xi'} D^\beta_{x'} p)) \kappa_{[\xi']} \|_{\mathcal{L}(H^{(\sigma_1, \sigma_2)}(\mathbb{R}), H^{(\sigma_1-\mu, \sigma_2)}(\mathbb{R}))} \\ &= \sup_{x', \xi'} \| [\xi']^{|\alpha|} \operatorname{op}_{x_n} (D^\alpha_{\xi'} D^\beta_{x'} p)\left(x', x_n/[\xi'], \xi', \xi_n[\xi']\right) \|_{\mathcal{L}(H^{(\sigma_1, \sigma_2)}(\mathbb{R}), H^{(\sigma_1-\mu, \sigma_2)}(\mathbb{R}))}. \end{aligned}$$

Since $D^\alpha_{\xi'} D^\beta_{x'} p$ is of order $\mu - |\alpha|$ we may assume that $\alpha = \beta = 0$. Now

$$\operatorname{op}_{x_n} p(x', x_n/[\xi'], \xi', \xi_n[\xi']) : H^{(\sigma_1, \sigma_2)}(\mathbb{R}) \to H^{(\sigma_1-\mu, \sigma_2)}(\mathbb{R}))$$

is continuous, and a bound for its norm is given by the suprema

$$\sup \left\{ |D^\alpha_{\xi_n} D^\beta_{x_n} \{ p(x', x_n/[\xi'], \xi', \xi_n[\xi']) \}| [\xi_n]^{-\mu} : x_n, \xi_n \in \mathbb{R} \right\}$$

for a finite number of derivatives. Since each of them is $O([\xi']^\mu)$ the proof is complete. ◁

2.8. Example: Multiplication operators.

(a) Multiplication M_{x_n} by x_n is an element of $S^{-1}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, H^s(\mathbb{R}_+), H^{s+(0,1)}(\mathbb{R}_+))$, $s \in \mathbb{R}^2$. We may replace the pair of Sobolev spaces by $(H^s_0(\overline{\mathbb{R}}_+), H^{s+(0,1)}_0(\overline{\mathbb{R}}_+))$.

(b) Let $\varphi \in \mathcal{C}^\infty_b(\overline{\mathbb{R}}_+)$ vanish to all orders at 0. Then multiplication M_φ by $\varphi(x_n)$ is an element of $S^{-\infty}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, H^s(\mathbb{R}_+), H^s(\mathbb{R}_+))$.

Proof. (a) follows from the fact that M_{x_n} has the symbol $a(x', \xi') = 1 \otimes M_{x_n}$ and that

$$\kappa^{-1}_{[\xi']} a(x', \xi')\, \kappa_{[\xi']u} = [\xi']^{-1} M_{x_n} u, \quad u \in \mathcal{C}^\infty_c(\mathbb{R}).$$

(b) follows from (a). ◁

2.9. Asymptotic summation and classical symbols. A sequence (a_j) of operator-valued symbols of orders $\mu - j$ with values in $\mathcal{L}(E,F)$ can be summed asymptotically to a symbol a of order μ, and a is unique modulo symbols of order $-\infty$.

A symbol a of order μ is said to be *classical,* if it has an asymptotic expansion $a \sim \sum_{j=0}^{\infty} a_j$ with a_j of order $\mu - j$ satisfying the homogeneity relation

$$a_j(y,\tilde{y},\lambda\eta) = \lambda^{\mu-j}\tilde{\kappa}_\lambda\, a_j(y,\tilde{y},\eta)\,\kappa_{\lambda^{-1}} \tag{2.7}$$

for all $\lambda \geq 1, |\eta| \geq R$ with a suitable constant R. We write $a \in S^{\mu}_{cl}(\mathbb{R}^q\times\mathbb{R}^q\times\mathbb{R}^q;E,F)$. For $E=\mathbb{C}^k,\ F=\mathbb{C}^l$ we recover the standard notion.

The key to many results on compositions is the lemma, below, which is adapted from Kumano-go [7, Chapter 2, Lemma 2.4].

2.10. Lemma. *Let $a \in S^{\mu}(\mathbb{R}^q\times\mathbb{R}^q\times\mathbb{R}^q;E,F)$. For $|\theta|\leq 1$ define a_θ by the oscillatory integral*

$$a_\theta(y,\eta) = \iint e^{-iz\zeta} a(y,y+z,\eta+\theta\zeta)dz đ\zeta. \tag{2.8}$$

Then the family $\{a_\theta : |\theta|\leq 1\}$ is uniformly bounded in S^μ; its seminorms can be estimated by those for a.

2.11. Theorem.

(a) *Let $a \in S^{\mu}(\mathbb{R}^q\times\mathbb{R}^q\times\mathbb{R}^q;E,F)$. Then there is a (unique) left symbol $a_L = a_L(y,\eta)$ for $\operatorname{op} a$ acting as in (2.4) and a (unique) right symbol $a_R = a_R(\tilde{y},\eta)$ acting as in (2.5). They are given by the oscillatory integrals*

$$a_L(y,\eta) = \iint e^{-iy\eta} a(y,y+z,\eta+\zeta)dz đ\zeta \text{ and} \tag{2.9}$$

$$a_R(\tilde{y},\eta) = \iint e^{i\tilde{y}\eta} a(\tilde{y}+z,\tilde{y},\eta+\zeta)dz đ\zeta \tag{2.10}$$

Moreover, we have

$$\begin{aligned} a_L(y,\eta) &= \sum_{|\alpha|<N}\frac{1}{\alpha!}\partial_\eta^\alpha D_{\tilde{y}}^\alpha a(y,\tilde{y},\eta)\Big|_{\tilde{y}=y} + N\sum_{|\gamma|=N}\int_0^1\frac{(1-\theta)^{N-1}}{\gamma!} \\ &\quad\times\iint e^{-iz\zeta}\partial_\eta^\gamma D_{\tilde{y}}^\gamma a(y,y+z,\eta+\theta\zeta)dz đ\zeta d\theta \text{ and} \end{aligned} \tag{2.11}$$

$$\begin{aligned} a_R(\tilde{y},\eta) &= \sum_{|\alpha|<N}\frac{(-1)^{|\alpha|}}{\alpha!}\partial_\eta^\alpha D_{y}^\alpha a(y,\tilde{y},\eta)\Big|_{y=\tilde{y}} + N\sum_{|\gamma|=N}\int_0^1\frac{(1-\theta)^{N-1}}{\gamma!} \\ &\quad\times\iint e^{+iz\zeta}\partial_\eta^\gamma D_{\tilde{y}}^\gamma a(y,y+z,\eta+\theta\zeta)dz đ\zeta d\theta \end{aligned} \tag{2.12}$$

with remainders $N\sum_{|\gamma|=N}\ldots$ in $S^{\mu-N}(\mathbb{R}^q\times\mathbb{R}^q;E,F)$.

(b) *Given $a \in S^{\mu}(\mathbb{R}^q\times\mathbb{R}^q;E,F)$ and $b \in S^{\tilde{\mu}}(\mathbb{R}^q\times\mathbb{R}^q;F,G)$ there is a left symbol $c \in S^{\mu+\tilde{\mu}}(\mathbb{R}^q\times\mathbb{R}^q;E,G)$ such that*

$$\operatorname{op} b \circ \operatorname{op} a = \operatorname{op} c.$$

As usual we write $c = b\#a$. We have the asymptotic expansion formula

$$\begin{aligned}(b\#a)(y,\eta) &= \sum_{|\alpha|<N}\frac{1}{\alpha!}\partial_\eta^\alpha b(y,\eta)D_y^\alpha a(y,\eta) + N\sum_{|\gamma|=N}\int_0^1\frac{(1-\theta)^{N-1}}{\gamma!}\\ &\quad\times\iint e^{+iz\zeta}\partial_\eta^\gamma b(y,\eta+\theta\zeta)D_y^\gamma a(y+z,\eta)\,dz\,đ\zeta\,d\theta \qquad (2.13)\end{aligned}$$

with a remainder in $S^{\mu+\tilde{\mu}-N}$.

Proof. The proof for existence and form of the left symbol is analogous to that of [7, Chapter 2, Theorem 2.5]. The right symbol is obtained by a simple modification. The estimates on the remainder follow from Lemma 2.10.

For the analysis of the composition let a_R be the right symbol for $\operatorname{op} a$. Then

$$\operatorname{op} b \circ \operatorname{op} a = \operatorname{op} b \circ \operatorname{op} a_R = \operatorname{op} \tilde{c}$$

with $\tilde{c}(y,\tilde{y},\eta) = b(y,\eta)a_R(\tilde{y},\eta)$. Choosing c as the left symbol of $\operatorname{op}\tilde{c}$ gives the assertion. Formula (2.13) follows from (2.11) and (2.12). For the scalar case see [7, Chapter 2, Theorems 2.6 and 3.1]. ◁

2.12. Duality. Let (E_-, E_0, E_+) be a triple of Hilbert spaces. We assume that all are embedded in a common vector space V and that $E_0 \cap E_+ \cap E_-$ is dense in $E_\pm$ as well as in E_0. Moreover we assume that there is a continuous, non-degenerate sesquilinear form $\langle\cdot,\cdot\rangle_E : E_+ \times E_- \to \mathbb{C}$ which coincides with the inner product of E_0 on $(E_+\cap E_0)\times(E_-\cap E_0)$. We ask that, via $\langle\cdot,\cdot\rangle_E$, we may identify E_+ with the dual of E_- and vice versa, and that

$$\|e\|'_{E_-} = \sup_{\|f\|_{E_+}=1}|\langle f,e\rangle_E|, \quad \|f\|'_{E_+} = \sup_{\|e\|_{E_-}=1}|\langle f,e\rangle_E|$$

furnish equivalent norms on E_- and E_+, respectively. Suppose there is a group action κ on V which has strongly continuous restrictions to E_0 and $E_\pm$, unitary on E_0, i.e., $\langle\kappa_\lambda e, f\rangle_E = \langle e, \kappa_{\lambda^{-1}} f\rangle_E$ for $e, f \in E_0$. Then

$$\langle\kappa_\lambda e, f\rangle_E = \langle e, \kappa_{\lambda^{-1}} f\rangle_E, \quad e \in E_+, f \in E_-,$$

since the identity holds on the dense set $(E_+\cap E_0)\times(E_-\cap E_0)$. In other words, the action κ on E_+ is dual to the action κ on E_- and vice versa.

Typical examples for the above situation are given by the triples of weighted Sobolev spaces

$$(H^{-\sigma}(\mathbb{R}), L^2(\mathbb{R}), H^\sigma(\mathbb{R})) \text{ and } (H_0^{-\sigma}(\overline{\mathbb{R}}_+), L^2(\mathbb{R}_+), H^\sigma(\mathbb{R}_+)), \quad \sigma \in \mathbb{R}^2.$$

Let (F_-, F_0, F_+) be an analogous triple of Hilbert spaces with group action $\tilde{\kappa}$, and let $a \in S^\mu(\mathbb{R}^q\times\mathbb{R}^q\times\mathbb{R}^q; E_-, F_-)$. We define a^* by $a^*(y,\tilde{y},\eta) = a(\tilde{y},y,\eta)^* \in \mathcal{L}(F_+, E_+)$, where the last asterisk denotes the adjoint operator with respect to the sesquilinear forms $\langle\cdot,\cdot\rangle_E$ and $\langle\cdot,\cdot\rangle_F$. It is not difficult to check that $a^* \in S^\mu(\mathbb{R}^q\times\mathbb{R}^q\times\mathbb{R}^q; F_+, E_+)$.

Moreover, we may introduce a continuous non-degenerate sesquilinear form

$$\langle\cdot,\cdot\rangle_{\mathcal{S}_E} : \mathcal{S}(\mathbb{R}^q, E_+)\times\mathcal{S}(\mathbb{R}^q, E_-) \to \mathbb{C}$$

by $\langle u,v\rangle_{\mathcal{S}_E} = \int\langle u(y),v(y)\rangle_E dy$. Analogously we define $\langle\cdot,\cdot\rangle_{\mathcal{S}_F}$.

The symbol a^* induces a continuous mapping $\operatorname{op} a^* : \mathcal{S}(\mathbb{R}^q, F_+) \to \mathcal{S}(\mathbb{R}^q, E_+)$. This is the unique operator satisfying

$$\langle(\operatorname{op} a^*)u, v\rangle_{\mathcal{S}_E} = \langle u, (\operatorname{op} a)v\rangle_{\mathcal{S}_F}.$$

2.13. Change of coordinates. Let $\chi : \mathbb{R}^q \to \mathbb{R}^q$ be a smooth diffeomorphism with all derivatives bounded and $0 < c \le |\det \chi(y)| \le C$ for all y.
Given an operator-valued pseudodifferential operator $A = \operatorname{op} a : \mathcal{S}(\mathbb{R}^q, E) \to \mathcal{S}(\mathbb{R}^q, F)$ we define its push-forward A^χ under χ by

$$(A^\chi(u \circ \chi))(y) = (Au)(\chi(y)).$$

For any lattice in $\mathbb{R}^q$ one finds functions $\{\varphi_j\}$, ψ_j, $j = 1, 2, \ldots$, each of them centered around a lattice point, such that $\sum \varphi_j = 1$ and $\varphi_j \psi_j = \varphi_j$.
Just as in the scalar case, cf. [7, Chapter 2, §6], it turns out that A^χ is an operator-valued pseudodifferential operator. In fact, modulo regularizing operators,

$$A^\chi = \operatorname{op} a^\chi$$

with the double symbol

$$\begin{aligned} a^\chi(y, y', \eta) = \sum_{j=1}^{\infty} \varphi_j(\chi(y))\, a(\chi(y), \chi(y'), \nabla_y \chi(y, y')^{-t} \eta)\, \psi_j(\chi(y')) \\ \times |\det \nabla_y \chi(y, y')|^{-1}\, |\det \partial_y \chi(y')|. \end{aligned} \tag{2.14}$$

Here,

$$\nabla_y \chi(y, y') = \int_0^1 \partial_y \chi(y + s(y - y')\, ds, \tag{2.15}$$

the superscript $-t$ means the inverse of the transpose, and the above formulas only make sense, if the lattice is sufficiently fine.
The leading term of the left symbol of a^χ is given by

$$a^\chi(y, y, \eta) = \sum \varphi(\chi(y)) a(\chi(y), \chi(y), \partial_y \chi(y)^{-t} \eta), \tag{2.16}$$

noting that $\nabla_y \chi(y, y) = \partial_y \chi(y)$.

2.14. Semiclassical operators. We shall now consider families $a(\hbar)$ of symbols with values in $\mathcal{L}(E, F)$, which depend smoothly on the parameter $\hbar \in (0, 1]$. We define an $\hbar$-scaling with the help of the group actions κ on E and $\tilde{\kappa}$ on F:

$$\begin{aligned} a_\hbar(\hbar; y, \tilde{y}, \eta) &= \tilde{\kappa}_\hbar^{-1} a(\hbar; y, \tilde{y}, \hbar\eta) \kappa_\hbar \quad \text{and} \\ a_{1/\hbar}(\hbar; y, \tilde{y}, \eta) &= \tilde{\kappa}_\hbar a(\hbar; y, \tilde{y}, \eta/\hbar) \kappa_\hbar^{-1}. \end{aligned}$$

Then:

(a) Let $a = a(\hbar)$, $\hbar \in (0, 1]$, be bounded in $S^\mu(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; E, F)$. Then so is $\{a_\hbar : \varepsilon \le \hbar \le 1\}$ for every $\varepsilon > 0$. Moreover, for every $\hbar_0 > 0$, the mapping $\hbar \mapsto a_\hbar$ is continuous in $\hbar_0$ with respect to the topology of S^μ.

(b) Let a be as above and write $a|_{\mathrm{diag}}(\hbar; y, \eta) = a(\hbar; y, y, \eta)$. Then

$$\begin{aligned} (a_\hbar)_L - (a|_{\mathrm{diag}})_\hbar &= \hbar\, b_L(\hbar) \quad \text{and} \\ (a_\hbar)_R - (a|_{\mathrm{diag}})_\hbar &= \hbar\, b_R(\hbar), \end{aligned}$$

with $b_{L,1/\hbar}$ and $b_{R,1/\hbar}$ bounded in the topology of $S^{\mu-1}$. The corresponding seminorms can be estimated in terms of the symbol seminorms for a.

(c) Given bounded families $a(\hbar)$ and $b(\hbar)$ in $S^\mu(\mathbb{R}^q\times\mathbb{R}^q;E,F)$ and $S^{\mu'}(\mathbb{R}^q\times\mathbb{R}^q;F,G)$, we find, for each N

$$b_\hbar\# a_\hbar-\sum_{|\alpha|<N}\frac{\hbar^{|\alpha|}}{\alpha!}(\partial_\eta^\alpha b)_\hbar(D_y^\alpha a)_\hbar=\hbar^N r_N(\hbar) \tag{2.17}$$

with $r_{N,1/\hbar}$ bounded in $S^{\mu+\mu'-N}(\mathbb{R}^q\times\mathbb{R}^q;E,G)$. The seminorms can be estimated in terms of those of a and b.

Proof. (a) The first assertion is immediate from (2.1) and the fact that, for $\varepsilon\le\hbar\le 1$, the quotient $[\hbar\eta]/[\eta]$ is both bounded and bounded away from zero. The second follows from the fact that

$$\begin{aligned}a_\hbar(\hbar;y,\eta)-a_{\hbar_0}(\hbar_0;y,\eta)&=(a_\hbar(\hbar;y,\eta)-a_\hbar(\hbar_0;y,\eta))+(a_\hbar(\hbar_0;y,\eta)-a_{\hbar_0}(\hbar_0;y,\eta))\\&=\left(\int_0^1\partial_\hbar a(\hbar_0+s(\hbar-\hbar_0);y,\hbar\eta)\,ds\;+\int_0^1\partial_\eta a(\hbar_0;y,\hbar_0\eta+\theta(\hbar-\hbar_0)\eta)\,d\theta\right)(\hbar-\hbar_0).\end{aligned}$$

(b) is immediate from (2.11) and (2.12), respectively, together with Lemma 2.10.
(c) In the expansion formula (2.13) let us replace there a by $a_\hbar$ and b by $b_\hbar$. We have $\partial_\eta^\alpha(b_\hbar)=\hbar^{|\alpha|}(\partial_\eta^\alpha b)_\hbar$ and $D_y^\alpha(a_\hbar)=(D_y^\alpha a)_\hbar$. This leads to the desired expansion. For the rescaled remainder $r^N_{1/\hbar}$ we obtain the expression in (2.13) with $\partial_\eta^\gamma b(y,\eta+\theta\zeta)$ replaced by $(\partial_\eta^\gamma b)(y,\eta+\hbar\theta\zeta)$. The boundedness in $S^{\mu+\mu'-N}$ then follows from Lemma 2.10. ◁

2.15. Corollary. *Let a be a pseudodifferential operator and χ a change of coordinates as in* 2.13. *It follows from* (2.14) *and* 2.14(b) *that*

$$(a_\hbar)^\chi-(a^\chi)_\hbar=\hbar\operatorname{op}b(\hbar)$$

with $b_{1/\hbar}$ bounded in the topology of $S^{\mu-1}$.

2.16. Lemma. *Let the supports of $\varphi,\psi\in\mathcal{C}_b^\infty(\mathbb{R}^q)$ have positive distance, and let $a\in S^\mu(\mathbb{R}^q\times\mathbb{R}^q;E,F)$. Then for any N we can write*

$$\varphi(\operatorname{op}a_\hbar)\psi=\hbar^N\operatorname{op}r_N(\hbar)$$

with a family r_N of symbols such that $r_{N,1/\hbar}$ is bounded in $S^{\mu-N}$.

Proof. This is immediate from the expansion formula (2.11). ◁

2.17. Lemma. *Let $a=a(\hbar)$, $\hbar\in(0,1]$, be bounded in $S^0(\mathbb{R}^q\times\mathbb{R}^q;L^2(\mathbb{R}_+),L^2(\mathbb{R}_+))$. Then also $a_\hbar$ is bounded in that class; its seminorms can be estimated in terms of those of a.*

Proof. Since κ is unitary on L^2, we only have to check that

$$D_\eta^\alpha D_y^\beta(a(\hbar;y,\hbar\eta))=O([\eta]^{-|\alpha|}).$$

This in turn is a consequence of the fact that

$$D_\eta^\alpha D_y^\beta(a(\hbar;y,\hbar\eta))=\hbar^{|\alpha|}(D_\eta^\alpha D_y^\beta a)(\hbar;y,\hbar\eta)$$

and that $\hbar[\eta][\hbar\eta]^{-1}$ is bounded. ◁

2.18. Proposition. *Let $s_1,s_2,t_1,t_2\ge 0$, $s_1+s_2>0$, $t_1+t_2>0$ and $m\in\mathbb{R}$. Moreover, let $g\in S^m(\mathbb{R}^q\times\mathbb{R}^q,H_0^{(-s_1,-t_1)}(\overline{\mathbb{R}}_+),H^{(s_2,t_2)}(\mathbb{R}_+))$ with $g(y,\eta)=0$ for large $|y|$. Then, for each $\varepsilon>0$, g can be approximated by elements in*

$$S^{-\infty}(\mathbb{R}^q\times\mathbb{R}^q,\mathcal{S}'(\mathbb{R}_+),\mathcal{S}(\mathbb{R}_+)),$$

which vanish for large $|y|$, in the topology of $S^{m+\varepsilon}=S^{m+\varepsilon}(\mathbb{R}^q\times\mathbb{R}^q,L^2(\mathbb{R}_+),L^2(\mathbb{R}_+))$.

For the proof we need the following well-known result:

2.19. Lemma. *Let $K \in \mathcal{K}(L^2(X))$ and $\varepsilon > 0$. Then there exist $\varphi_1, \dots, \varphi_N, \psi_1, \dots, \psi_N \in \mathcal{C}_c^\infty(X^\circ)$ such that*

$$\|K - \sum_{j=1}^N \varphi_j \otimes \psi_j\|_{L^2(X)} < \varepsilon.$$

Proof of Proposition 2.18. By composing with the operator $[\eta]^{-m-\varepsilon} \otimes \mathrm{id}$, we may assume that $m = -\varepsilon$. Choose $\varphi \in \mathcal{C}_c^\infty([0,\infty))$ with $\varphi(t) \equiv 1$ for $t \le 1$, and let

$$g_N(y,\eta) = g(y,\eta)\varphi(|\eta|/N), \quad N \in \mathbb{N}.$$

Then g_N is a regularizing symbol with values in $\mathcal{L}(H_0^{(-s_1,-t_1)}(\overline{\mathbb{R}}_+), H^{(s_2,t_2)}(\mathbb{R}_+))$. Moreover, g_N tends to g in the topology of S^0. It is therefore sufficient to approximate g_N in S^0. As g_N vanishes for (y,η) outside a compact set, we have in fact

$$g_N \in \mathcal{S}(\mathbb{R}^q \times \mathbb{R}^q)\hat{\otimes}_\pi \mathcal{L}(H_0^{(-s_1,-t_1)}(\overline{\mathbb{R}}_+), H^{(s_2,t_2)}(\mathbb{R}_+)) \hookrightarrow \mathcal{S}(\mathbb{R}^q \times \mathbb{R}^q)\hat{\otimes}_\pi \mathcal{K}(L^2(\mathbb{R}_+)).$$

According to Lemma 2.19, each element of $\mathcal{K}(L^2(\mathbb{R}_+))$ can be approximated in $\mathcal{L}(L^2(\mathbb{R}_+))$ by an integral operator with a rapidly decreasing kernel. Each of these defines a continuous operator $\mathcal{S}'(\mathbb{R}_+) \to \mathcal{S}(\mathbb{R}_+)$. Hence g_N can be approximated, in the topology of

$$\mathcal{S}(\mathbb{R}^q \times \mathbb{R}^q)\hat{\otimes}_\pi \mathcal{L}(L^2(\mathbb{R}_+)) \hookrightarrow S^{-\infty}(\mathbb{R}^q \times \mathbb{R}^q; L^2(\mathbb{R}_+), L^2(\mathbb{R}_+))$$

by elements in the tensor product $\mathcal{S}(\mathbb{R}^q \times \mathbb{R}^q)\hat{\otimes}_\pi \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}_+)$, hence also by elements of $S^{-\infty}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{S}'(\mathbb{R}_+), \mathcal{S}(\mathbb{R}_+))$ which vanish for y outside a compact set. $\triangleleft$

2.20. Remark. A symbol g in $S^{-\infty}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{S}'(\mathbb{R}_+), \mathcal{S}(\mathbb{R}_+))$ which vanishes for y outside a compact set induces an operator $\mathrm{op}\, a$ on $L^2(\mathbb{R}_+^{q+1})$ with an integral kernel which is rapidly decreasing and thus can be approximated by a smooth compactly supported integral kernel. This slightly improves the statement of Proposition 2.18.

3. Semiclassical Operators in Boutet de Monvel's Calculus

3.1. Pseudodifferential operators. Let $p \in S^\mu(\mathbb{R}^n \times \mathbb{R}^n)$ and $u \in \mathcal{S}(\mathbb{R})$. Then

$$\begin{aligned}
\mathrm{op}_{x_n}(p_\hbar)u(x_n) &= \int\int_0^\infty e^{i(x_n-y_n)\xi_n} p(x', x_n, \hbar\xi', \hbar\xi_n)\ u(y_n)\, dy_n đ\xi_n \\
&= \int\int_0^\infty e^{i(x_n/\hbar-y_n)\xi_n} p(x', x_n, \hbar\xi', \xi_n)\ u(\hbar y_n)\, dy_n đ\xi_n \\
&= \kappa_{\hbar^{-1}} \int\int_0^\infty e^{i(x_n-y_n)\xi_n} p(x', \hbar x_n, \hbar\xi', \xi_n)\kappa_\hbar u(y_n)\, dy_n đ\xi_n.
\end{aligned}$$

In case p is independent of x_n, the last term equals $\left(\mathrm{op}_{x_n} p\right)_\hbar u(x_n)$ where the subscript indicates that we use the $\hbar$-scaled symbol associated to the operator-valued symbol $\mathrm{op}_{x_n} p$.

3.2. Potential, trace, and singular Green boundary symbol operators. We define the $\hbar$-scaled operators as in 2.14, noting that the group action on $\mathbb{C}$ is the identity. Specifically,

$$\begin{aligned}
k_\hbar(x',\xi') &= \kappa_\hbar^{-1} k(x', \hbar\xi') && \text{(potential symbols)} \\
t_\hbar(x',\xi') &= t(x', \hbar\xi')\kappa_\hbar && \text{(trace symbols)} \\
g_\hbar(x',\xi') &= \kappa_\hbar^{-1} g(x', \hbar\xi')\kappa_\hbar && \text{(singular Green symbols).}
\end{aligned}$$

3.3. Lemma. *Let $g \in S^\mu(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}'(\mathbb{R}_+), \mathcal{S}(\mathbb{R}_+))$, $\mu \in \mathbb{R}$, and $\varphi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$, supported in $\mathbb{R}^n_+$. Then, for every $N \in \mathbb{N}$, we have*

$$\varphi \operatorname{op}(g_\hbar) = \hbar^N \operatorname{op} r_N(\hbar)$$

with a family r_N of singular Green symbols such that $r_{N,1/\hbar}$ is bounded in $S^{\mu-N}$.

Proof. Write $\varphi_N(x) = x_n^{-N}\varphi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$. In view of the fact that $M_{x_n^N}\kappa_\hbar^{-1} = \hbar^N \kappa_\hbar^{-1} M_{x_n^N}$ we have

$$M_\varphi \operatorname{op} g_\hbar = \hbar^N M_{\varphi_N} \operatorname{op}(\kappa_\hbar^{-1}(x_n^N g(x', \hbar\xi')\kappa_\hbar).$$

This shows the assertion, since $x_n^N g$ is a singular Green symbol of order $\mu - N$. ◁

3.4. Lemma. *Let g_1 and g_2 be singular Green symbols of orders μ_1 and μ_2, respectively. Then*

$$g_{1,\hbar} g_{2,\hbar} - (g_1 g_2)_\hbar = \hbar\; c(\hbar)$$

for a family $c(\hbar)$, $0 < \hbar \le 1$, of singular Green symbols with $c_{1/\hbar}$ bounded of order $\mu_1 + \mu_2 - 1$.

Proof. This is immediate from 2.14(c) in connection with Definition 3.2. ◁

3.5. Remark. In the following Proposition we will show an analog of 2.14(c). The statement becomes more involved, since we have to take care of the leftover term. We thus fix the notation beforehand.

Let $p_j \in S^{\mu_j}_{\mathrm{tr}}(\mathbb{R}^n \times \mathbb{R}^n)$, $\mu_j \in \mathbb{Z}$, $j = 1, 2$. For fixed (x', ξ'), we consider the operators $(\operatorname{op}^+_{x_n} p_j)(x', \xi')$. Their composition gives rise to a (x', ξ')-dependent leftover term. We denote by $l(p_1, p_2)$ its operator-valued singular Green symbol. In order to study it we follow Grubb [6, (2.6.18)ff] and decompose as in (1.1):

$$p_j = s_j + q_j \tag{3.1}$$

with differential symbols s_j and symbols q_j of normal order -1. We write $s_2(x, \xi) = \sum_{j=0}^{\mu_2} a_j(x, \xi')\xi_n^j$ and $\gamma_m = \gamma_0 \circ D_n^m$ Then

$$l(p_1, p_2) = \sum_{m=0}^{\mu_2 - 1} k_m \gamma_m + g^+(q_1) g^-(q_2). \tag{3.2}$$

with the potential symbol k_m given by

$$k_m(x', \xi')v = ir^+ \left(p_1(x, \xi', D_n) \sum_{l=m+1}^{\mu_2} a_l(x, \xi') D_n^{l-1-m} \right) (v \otimes \delta); \tag{3.3}$$

and the singular Green symbols $g^\pm$ defined by

$$g^+(q_1) = r^+ \operatorname{op}_{x_n}(q_1) e^- J \quad \text{and} \quad g^-(q_2) = J r^- \operatorname{op}_{x_n}(q_2) e^+ \tag{3.4}$$

with the reflection operator $J : f(x', x_n) \mapsto f(x', -x_n)$.

We will need a semiclassical version of the following well-known statement:

3.6. Theorem. *Given a pseudodifferential symbol p of order μ with the transmission property and $l \in \mathbb{N}_0$, the prescription*

$$v \mapsto r^+ \operatorname{op} p(v \otimes D_n^l \delta), \quad v \in \mathcal{S}(\mathbb{R}^{n-1}),$$

defines a potential symbol k of order $\mu+l+1/2$ whose symbol seminorms can be estimated in terms of those of p. Writing $p=p^0+x_np^1$ with $p^0(x',\xi)=p(x',0,\xi)$, we have $k=k^0+k^1$, where

$$k^0(x',\xi')=\mathrm{op}_{x_n}(p^0(x',\xi)\xi_n^l)(\delta_{x_n=0}), \tag{3.5}$$

considered as a multiplication operator on $\mathcal{S}(\mathbb{R}_+)$, and k^1 is of order $\mu+l-1/2$.

For a proof in the spirit of operator-valued symbols see [13, Lemma 2.11]. In the semi-classical situation we obtain:

3.7. Lemma. *Under the above assumptions,*

$$v\mapsto r^+\,\mathrm{op}_{x_n}\,p_\hbar(v\otimes\hbar^{l+1/2}D_n^l\delta),\quad v\in\mathcal{S}(\mathbb{R}^{n-1})$$

defines a potential boundary symbol operator $k(\hbar)$. We have

$$k(\hbar)=k^0_\hbar+\hbar k^1(\hbar)$$

with k^0 defined in (3.5) and $k^1_{1/\hbar}$ bounded of order $\mu+l-1/2$.

Proof. Replacing p by $p\,\xi_n^l$ we can assume $l=0$. This yields the potential symbol

$$\begin{aligned}
k(\hbar;x',\xi') &= \hbar^{1/2}r^+\int e^{ix_n\xi_n}p(x,\hbar\xi)\,đ\xi_n\\
&= \hbar^{-1/2}r^+\int e^{ix_n\xi_n/\hbar}\left(p^0(x',\hbar\xi',\xi_n)+x_np^1(x,\hbar\xi',\xi_n)\right)\,đ\xi_n\\
&= \kappa_\hbar^{-1}r^+\left((\mathrm{op}_{x_n}p^0)(x',\hbar\xi')\delta+\int e^{ix_n\xi_n}\hbar x_np^1(x,\hbar x_n,\hbar\xi',\xi_n)\,đ\xi_n\right)\\
&= k^0_\hbar(x',\xi')+\hbar k^1(\hbar;x',\xi').
\end{aligned}$$

Rescaling the potential symbols $k^1(\hbar)$ we obtain

$$k^1_{1/\hbar}(\hbar;x',\xi')=\int e^{ix_n\xi_n}x_np^1(x,\hbar x_n,\xi',\xi_n)\,đ\xi_n=\int e^{ix_n\xi_n}(-D_{\xi_n})p^1(x,\hbar x_n,\xi',\xi_n)\,đ\xi_n,$$

which is uniformly bounded in $S^{\mu-1/2}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathbb{C},\mathcal{S}(\mathbb{R}_+))$ by Theorem 3.6. ◁

In the following proposition, we shall analyze the relation between the interior symbols and the leftover term.

3.8. Proposition. *We use the notation of Remark 3.5. Then*

$$\mathrm{op}^+_{x_n}p_{1,\hbar}\circ\mathrm{op}^+_{x_n}p_{2,\hbar}-\mathrm{op}^+_{x_n}(p_1p_2)_\hbar-l(p_1,p_2)_\hbar=\hbar\,\left(\mathrm{op}^+_{x_n}c(\hbar)+d(\hbar)\right)$$

for two families c and d with $c_{1/\hbar}$ bounded in $S^{\mu_1+\mu_2-1}_{\mathrm{tr}}(\mathbb{R}^n\times\mathbb{R}^n)$ and $d_{1/\hbar}$ bounded in $S^{\mu_1+\mu_2-1}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{S}'(\mathbb{R}_+),\mathcal{S}(\mathbb{R}_+))$.

Proof. We already know from 2.14 (for the standard $\mathbb{C}$-valued case) that

$$\mathrm{op}_{x_n}p_{1,\hbar}\,\mathrm{op}_{x_n}p_{2,\hbar}-\mathrm{op}_{x_n}(p_1p_2)_\hbar=\hbar\ \mathrm{op}(c(\hbar)),$$

with $c_{1/\hbar}$ bounded in $S^{\mu_1+\mu_2-1}$. In fact, we even obtain boundedness in the corresponding class with the transmission property, since we have an asymptotic expansion for c with terms bounded in the topology of symbols with the transmission property.

Let us next consider the symbol $l(p_{1,\hbar},p_{2,\hbar})$ of the leftover term $L(\mathrm{op}_{x_n}p_{1,\hbar},\mathrm{op}_{x_n}p_{2,\hbar})$, which we will compute according to (3.2), (3.3). Replacing ξ by $\hbar\xi$, we obtain

$$l(p_{1,\hbar},p_{2,\hbar})=\sum_{m=0}^{\mu_2-1}k_m(\hbar)\gamma_m+g^+(q_{1,\hbar})g^-(q_{2,\hbar}).$$

Using the notation introduced above, the potential symbol k_m is given by

$$k_m(\hbar; x', \xi')v = ir^+ \left(\mathrm{op}_{x_n} p_1(x, \hbar\xi) \sum_{l=m+1}^{\mu_2} a_l(x, \hbar\xi')\hbar^l D_n^{l-1-m} \right) (v \otimes \delta).$$

By Lemma 3.7, $\hbar^{-m-1/2} k_m(\hbar)$ is a family of potential symbols of order $\mu_1 + \mu_2 - m - 1/2$, equal to $k^0_{m,\hbar}$ modulo lower order terms. In view of the fact that $\gamma_{m,\hbar} = \hbar^{m+1/2}\gamma_m$, this shows that the composition $k_m(\hbar)\gamma_m$ is a singular Green symbol which equals $k^0_{m,\hbar}\gamma_{m,\hbar}$ modulo lower order terms of the desired form.

The singular Green symbols $g^\pm(p) = g^\pm(p)(x', \xi')$ associated to a pseudodifferential symbol p of negative normal order are the integral operators with the kernels

$$\tilde{g}^\pm(p)(x', \xi', x_n, y_n) = \int e^{iz\xi_n} p(x', x_n, \xi', \xi_n) đ\xi_n \Big|_{z=\pm(x_n+y_n)} \tag{3.6}$$

so that

$$\tilde{g}^\pm(p_\hbar)(x', \xi', x_n, y_n) = \hbar^{-1} \int e^{iz\xi_n} p(x', x_n, \hbar\xi', \xi_n) đ\xi_n \Big|_{z=\pm(x_n+y_n)/\hbar}$$

This implies that

$$g^\pm(p_\hbar)(x', \xi') = \kappa_\hbar^{-1} g^\pm(p(x', \hbar x_n, \hbar\xi', \xi_n))\kappa_\hbar$$

Writing $p = p^0 + x_n p^1$ as before, we have

$$g^\pm(p_\hbar)(x', \xi') = \left(g^\pm(p^0)\right)_\hbar + \hbar\kappa_\hbar^{-1} x_n g^\pm\left(p^1(x', \hbar x_n, \hbar\xi', \xi_n)\right)\kappa_\hbar$$

As $p^1(x', \hbar x_n, \hbar\xi', \xi_n)$, $0 < \hbar \le 1$, is a bounded family of pseudodifferential operators of order μ with the transmission property and since multiplication by x_n lowers the order by 1 according to 2.8(a), we obtain the assertion of the proposition by first applying this consideration to $g^+(q_{1,\hbar})$ and $g^-(q_{2,\hbar})$ and then using Lemma 3.4. $\triangleleft$

4. The Graph Projection

4.1. Definition. The graph projection of a bounded operator a on a Hilbert space H is the operator $\mathcal{G}(a)$ on $H \oplus H$ given by

$$\mathcal{G}(a) = \begin{pmatrix} (1 + a^*a)^{-1} & (1 + a^*a)^{-1}a^* \\ a(1 + a^*a)^{-1} & a(1 + a^*a)^{-1}a^* \end{pmatrix} = \begin{pmatrix} (1 + a^*a)^{-1} & (1 + a^*a)^{-1}a^* \\ a(1 + a^*a)^{-1} & 1 - (1 + aa^*)^{-1} \end{pmatrix}.$$

For unbounded operators, it is not a priori clear that the above definition makes sense, nor is the second identity obvious. We will now have a closer look.

4.2. The framework. Let $V \hookrightarrow H \hookrightarrow V'$ be Hilbert spaces, with V dense in H, and assume that V' is the dual space of V with respect to an extension $\langle\cdot,\cdot\rangle$ of the inner product in H. Moreover let $a_0 : V \to H$ be a bounded operator with adjoint $a_0^* : H \to V'$. We assume that a_0 is closable in H and denote by a the closure. Explicitly: The domain $\mathcal{D}$ of a consists of all x in H for which there exists a sequence (x_n) in V with $x_n \to x$ in H and $a_0 x_n \to y$ in H for some $y \in H$. In that case, we define $ax = y$.

4.3. Lemma. *$\mathcal{D}$ naturally is a Hilbert space with the inner product*

$$\langle\langle x, y\rangle\rangle_{\mathcal{D}} = \langle x, y\rangle + \langle ax, ay\rangle. \tag{4.7}$$

V is dense in $\mathcal{D}$.

Proof. Clearly, (4.7) defines an inner product on $\mathcal{D}$. The associated norm $\|x\|_{\mathcal{D}} = (\|x\|^2 + \|ax\|^2)^{1/2}$ is the graph norm with respect to which $\mathcal{D}$ is complete. Hence $\mathcal{D}$ is a Hilbert space. V is dense in $\mathcal{D}$ by construction. $\triangleleft$

4.4. Lemma. *Let $x \in H$. Then the element $a_0^* x \in V'$ extends to a continuous linear functional on $\mathcal{D}$, and*

$$\langle a_0^* x, y\rangle = \langle x, ay\rangle, \quad x \in H, y \in \mathcal{D}. \tag{4.8}$$

Proof. For v in the dense subspace V we have

$$|\langle a_0^* x, v\rangle| = |\langle x, av\rangle| \le \|x\|\|av\| \le \|x\|\|v\|_{\mathcal{D}}$$

so that $a_0^* x$ extends continuously to $\mathcal{D}$. The stated identity follows. $\triangleleft$

We now denote by $\mathcal{E}$ the range of the operator $1 + a_0^* a : \mathcal{D} \to V'$.

4.5. Lemma. *The elements of $\mathcal{E}$ define continuous linear functionals on $\mathcal{D}$. The norm of $(1 + a_0^* a)x$ on $\mathcal{D}$ is $\|x\|_{\mathcal{D}}$, and the operator $1 + a_0^* a$ is injective.*

Proof. As $\mathcal{D} \hookrightarrow H$, the first statement follows from Lemma 4.4. Note that

$$\begin{aligned}\|(1 + a_0^* a)x\|_{\mathcal{L}(\mathcal{D},\mathbb{C})} &= \sup_{\|y\|_{\mathcal{D}} \le 1} |\langle (1 + a_0^* a)x, y\rangle| = \sup_{\|y\|_{\mathcal{D}} \le 1} |\langle x, y\rangle + \langle ax, ay\rangle| \\ &= \sup_{\|y\|_{\mathcal{D}} \le 1} |\langle\langle x, y\rangle\rangle_{\mathcal{D}}| = \|x\|_{\mathcal{D}}.\end{aligned}$$

This implies injectivity. $\triangleleft$

4.6. Lemma. *$\mathcal{E}$ inherits a Hilbert space structure from $\mathcal{D}$. The associated norm is the norm in $\mathcal{D}' = \mathcal{L}(\mathcal{D}, \mathbb{C})$. This allows us to identify $\mathcal{E}$ with the dual of $\mathcal{D}$ with respect to the sesquilinear pairing between V and V'. $\mathcal{E}$ contains the range of a_0^**

Proof. Let $e = x + a_0^* ax$ and $f = y + a_0^* ay$ be two elements of $\mathcal{E}$. Then we let

$$\langle\langle e, f\rangle\rangle_{\mathcal{E}} = \langle\langle x, y\rangle\rangle_{\mathcal{D}}, \tag{4.9}$$

so that the Hilbert space structure of $\mathcal{D}$ carries over to $\mathcal{E}$. The associated norm of $e = (1 + a_0^* a)x$ then is equal to $\|x\|_{\mathcal{D}}$, which is the norm of $(1 + a_0^* a)x$ in $\mathcal{D}'$ by Lemma 4.5. Moreover, the identity

$$\langle x + a_0^* ax, y\rangle = \langle\langle x, y\rangle\rangle_{\mathcal{D}}$$

shows that the action of $\mathcal{E}$ on $\mathcal{D}$ via the sesquilinear pairing coincides with the pairing with elements of $\mathcal{D}$, which gives the whole dual space. By Lemma 4.4, $a_0^*(H) \subset \mathcal{D}' = \mathcal{E}$. $\triangleleft$

4.7. Corollary. *The operator $1 + a_0^* a : \mathcal{D} \to \mathcal{E}$ is invertible with a bounded inverse. Moreover, the operators $(1+a_0^* a)^{-1}$, $(1+a_0^* a)^{-1} a_0^*$, $a(1+a_0^* a)^{-1}$, and $a(1+a_0^* a)^{-1} a_0^*$ define elements of $\mathcal{L}(H)$.*

4.8. Lemma. *The restriction of $(1 + a_0^* a)^{-1}$ to H is self-adjoint, and the operators $a(1 + a_0^* a)^{-1}$ and $(1 + a_0^* a)^{-1} a_0^*$ are adjoints of each other in $\mathcal{L}(H)$.*

Proof. Let $x \in H \subset \mathcal{E}$ and $z = (1 + a_0^* a)^{-1} x \in \mathcal{D}$. For $y \in H$, Lemma 4.4 implies that

$$\langle (1 + a_0^* a)^{-1} y, x\rangle = \langle (1 + a_0^* a)^{-1} y, (1 + a_0^* a)z\rangle = \langle y, z\rangle = \langle y, (1 + a_0^* a)^{-1} x\rangle.$$

This shows that $(1 + a_0^* a)^{-1}$ is selfadjoint. For $x, y \in H$ the element $a_0^* x \in \mathcal{E}$ has a preimage z in $\mathcal{D}$ under $1 + a_0^* a$. We infer from (4.8) that

$$\begin{aligned}\langle a(1 + a_0^* a)^{-1} y, x\rangle &= \langle (1 + a_0^* a)^{-1} y, a_0^* x\rangle = \langle (1 + a_0^* a)^{-1} y, (1 + a_0^* a)z\rangle \\ &= \langle y, z\rangle = \langle y, (1 + a_0^* a)^{-1} a_0^* x\rangle.\end{aligned}$$

This shows the second statement. $\triangleleft$

4.9. Notation. In the above we wrote a_0^* in order to stress the fact that this operator is *not* the H-valued Hilbert space adjoint of a but an operator with values in V'. Now that this has been made clear we shall go back to the simpler notation and write a^*.

4.10. The set-up. In the sequel,

$$T : \mathcal{C}^\infty(X, E_1) \to \mathcal{C}^\infty(X, E_2)$$

will be a Fredholm operator of order and class zero in Boutet de Monvel's calculus. Following an idea of Elliott-Natsume-Nest [4] we will associate to T the operator $A = \Lambda_-^{m,+}T$ for some $m > n$ and then study the graph projection.

The operator T induces a Fredholm operator $T_m : H^m(X, E_1) \to H^m(X, E_2)$. Its adjoint $T_m^* : H_0^{-m}(X, E_2) \to H_0^{-m}(X, E_1)$ extends the L^2-adjoint T^*. We let

$$A = \Lambda_-^{m,+}T_m : H^m(X, E_1) \to L^2(X, E_2).$$

This is an operator of order m and class zero; there is no leftover term in the composition. The adjoint A^* is the operator $(\Lambda_-^{m,+}T_m)^* = T_m^*\Lambda_+^m : L^2(X, E_2) \to H_0^{-m}(X, E_1)$.

We consider the composition A^*A as the bounded operator

$$T_m^*\Lambda_+^m\Lambda_-^{m,+}T_m : H^m(X, E_1) \to H_0^{-m}(X, E_1).$$

On the other hand, $A = \Lambda_-^{m,+}T$ extends to a bounded operator from $L^2(X, E_1)$ to $H^{-m}(X, E_2)$ with adjoint $A^* = T^*\Lambda_+^m : H_0^m(X, E_2) \to L^2(X, E_1)$, and $AA^* = \Lambda_-^{m,+}TT^*\Lambda_+^m$ maps $H_0^m(X, E_2)$ to $H^{-m}(X, E_2)$.

4.11. Lemma. *We have natural embeddings*

$$\begin{aligned} &H^m(X, E_j) \hookrightarrow L^2(X, E_j) \hookrightarrow H_0^{-m}(X, E_j) \quad \textit{and} \\ &H_0^m(X, E_j) \hookrightarrow L^2(X, E_j) \hookrightarrow H^{-m}(X, E_j), \end{aligned}$$

$j = 1, 2,$ *and topological isomorphisms*

$$1 + A^*A : H^m(X, E_1) \to H_0^{-m}(X, E_1) \quad \textit{and} \quad 1 + AA^* : H_0^m(X, E_2) \to H^{-m}(X, E_2).$$

Proof. The embeddings are well-known. The second statement follows from Corollary 4.7, applied to the operator

$$a = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} : \begin{matrix} H^m(X, E_1) \\ \oplus \\ H_0^m(X, E_2) \end{matrix} \to \begin{matrix} L^2(X, E_1) \\ \oplus \\ L^2(X, E_2) \end{matrix},$$

together with the fact that – due to elliptic regularity – the domain of the closure of A is $H^m(X, E_1)$, that of A^* is $H_0^m(X, E_2)$. ◁

As a consequence, the operators $A(1 + A^*A)^{-1}A^*$ and $(1 + AA^*)^{-1}$ are bounded operators on $L^2(X, E_1)$ and $L^2(X, E_2)$, respectively.

4.12. Lemma. *The restriction of AA^* to $H_0^{2m}(X, E_2)$ maps to $L^2(X, E_2)$.*

Proof. We know that $\Lambda_+^m : H_0^{2m} \xrightarrow{\cong} H_0^m$. Now we observe that H_0^m naturally embeds into H^m, and thus TT^* defines a bounded map from H_0^m to H^m. Finally, $\Lambda_-^m : H^m \xrightarrow{\cong} L^2$. ◁

4.13. Lemma. $A(1 + A^*A)^{-1}A^* = 1 - (1 + AA^*)^{-1}$ *on* $L^2(X, E_2)$.

Proof. According to Lemma 4.12, the composition $A(1+A^*A)^{-1}A^*(1+AA^*)$ is defined as a bounded operator from H_0^{2m} to L^2, and we have

$$A(1+A^*A)^{-1}A^*(1+AA^*) = A(1+A^*A)^{-1}(1+A^*A)A^* = AA^*. \tag{4.10}$$

We next denote by $\mathcal{R}$ the range of the restriction of $(1+AA^*)^{-1}$ to L^2, so that $1+AA^*:\mathcal{R}\to L^2$ is an isomorphism. According to 4.11 and 4.12, we have $H_0^{2m}\subseteq\mathcal{R}\subseteq H_0^m$. As (4.10) extends to $\mathcal{R}$, the compositions, below, are defined on L^2, and

$$\begin{aligned}&A(1+A^*A)^{-1}A^*\\ &\quad= A(1+A^*A)^{-1}A^*(1+AA^*)(1+AA^*)^{-1} = AA^*(1+AA^*)^{-1} = 1-(1+AA^*)^{-1}.\end{aligned}$$

◁

4.14. Lemma. *$\mathcal{G}(A)$ and $e=\begin{pmatrix}0&0\\0&1\end{pmatrix}$ define idempotents in $\mathcal{K}(L^2(X,E_1\oplus E_2))^\sim$, where as usual the tilde indicates the unitization.*
In particular, the difference $[\mathcal{G}(A)]-[e]$ defines a class in $K_0(\mathcal{K}(L^2(X,E_1\oplus E_2)))$.

Proof. Apart from the one in the lower right corner, the entries in $\mathcal{G}(A)$ are compact on L^2 as a consequence of the compact embeddings $H_0^m\hookrightarrow L^2$ and $H^m\hookrightarrow L^2$. By Lemma 4.13, the last entry differs from the identity by the compact operator $(1+AA^*)^{-1}$. ◁

4.15. Theorem. *The class $[\mathcal{G}(A)]-[e]$ in $K_0(\mathcal{K}(L^2(X,E_1\oplus E_2)))$ equals*

$$[\pi_{\ker A}]-[\pi_{\ker A^*}] = [\pi_{\ker T}]-[\pi_{\ker T^*}]\,.$$

Here, π_V denotes the orthogonal projection onto V with respect to the L^2-inner product.

Proof. Replacing A by tA for $t\ge 1$, we consider $\mathcal{G}(tA)$, which is a norm continuous family of idempotents. We claim that $\mathcal{G}(tA)$ converges to

$$\begin{pmatrix}\pi_{\ker T} & 0\\ 0 & 1-\pi_{\ker T^*}\end{pmatrix}$$

as $t\to\infty$. We let $\mathcal{H}_1=(1+A^*A)^{-1}(L^2)$ and $\mathcal{H}_2=.(1+AA^*)^{-1}(L^2)$. This allows us to consider A^*A and AA^* as an unbounded operators on L^2 with domains $\mathcal{H}_1$ and H_2, respectively. The graph projection is not affected by this change. For the unbounded operator, however, the statements are well-known. They are a consequence of the fact that 0 is an isolated point in the spectrum. Alternatively, the statement can be checked by a direct computation. ◁

4.16. Corollary. *For $m>n$,*

$$\operatorname{ind} T = \operatorname{Tr}(1+A^*A)^{-1} - \operatorname{Tr}(1+AA^*)^{-1}.$$

Proof. The difference $\mathcal{G}(A)-e$ is a trace class operator in $\mathcal{L}(L^2)$ since its four entries are trace class operators. This in turn follows from the fact that the embeddings $H^m\hookrightarrow L^2$ and $L^2\hookrightarrow H_0^{-m}$ are trace class. According to Theorem 4.15 we then have

$$\operatorname{ind} A = \operatorname{Tr}([\pi_{\ker A}]-[\pi_{\operatorname{coker} A}]) = \operatorname{Tr}(\mathcal{G}(A)-e) = \operatorname{Tr}(1+A^*A)^{-1} - \operatorname{Tr}(1+AA^*)^{-1}.$$

◁

5. The Graph Projections of the Symbols

5.1. Notation. We denote by

$$p^m = \sigma_\psi^m(A) \quad \text{and} \quad c^m = \sigma_\partial^m(A)$$

the homogeneous principal pseudodifferential symbol and the homogeneous principal boundary symbol of the operator A in 4.10. Both are invertible.

Locally

$$c^m(x',\xi') = p^{m,+}(x',0,\xi',D_n) + g^m(x',\xi') \tag{5.11}$$

with a suitable strictly homogeneous singular Green part g^m. We then choose a smooth function p on T^*X which coincides with p^m for $|\xi| \ge 1$ and a smooth singular Green symbol g which coincides with g^m for $|\xi'| \ge 1$ and let

$$c(x',\xi') = p^+(x',0,\xi',D_n) + g(x',\xi').$$

We next apply Corollary 4.7 to the operator family

$$1 + c^*(x',\xi')c(x',\xi') : H^m(\mathbb{R}_+, \widetilde{E}_1) \to H_0^{-m}(\mathbb{R}_+, \widetilde{E}_1), \quad (x',\xi')? \in T^*X.$$

For each (x',ξ'), we denote by $\mathcal{D}_{(x',\xi')}$ the domain of the closure and by $\mathcal{E}_{(x',\xi')}$ the range.

5.2. Proposition. *For each choice of (x',ξ') we have*

$$\mathcal{D}_{(x',\xi')} = H^m(\mathbb{R}_+, \widetilde{E}_1)) \quad \textit{and} \quad \mathcal{E}_{(x',\xi')} = H_0^{-m}(\overline{\mathbb{R}}_+, \widetilde{E}_1)). \tag{5.12}$$

The operator family

$$1 + c^*(x',\xi')c(x',\xi') : H^m(\mathbb{R}_+, \widetilde{E}_1) \to H_0^m(\overline{\mathbb{R}}_+, \widetilde{E}_1)$$

is pointwise invertible.

The inverse $(1+c^*c)^{-1}$ is an element of

$$S^{-2m}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; H_0^{-m}(\mathbb{R}_+, \widetilde{E}_1), H^m(\mathbb{R}_+, \widetilde{E}_1)). \tag{5.13}$$

We shall improve this result in 5.3, below.

Proof. By definition, $\mathcal{D}_{(x',\xi')} \subseteq H^m(\mathbb{R}_+, \widetilde{E}_1)$ is the domain of the closure of the operator

$$c(x',\xi') : H^m(\mathbb{R}_+, \widetilde{E}_1) \to L^2(\mathbb{R}_+, \widetilde{E}_2)$$

in $L^2(\mathbb{R}_+, \widetilde{E}_1)$. It consists of all $v \in L^2$ for which there is a sequence v_k in H^m with $v_k \to v$ in L^2 and $c(x',\xi')v_k \to w$ in L^2 for some w which then is defined to be $c(x',\xi')v$.

As $g(x',\xi') : \mathcal{S}'(\mathbb{R}_+, \widetilde{E}_1) \to \mathcal{S}(\mathbb{R}_+, \widetilde{E}_2)$ is continuous, $g(x',\xi')v_k$ will converge for any convergent L^2-sequence (v_k); hence the domain is independent of g. We therefore have

$$\mathcal{D}_{(x',\xi')} \subseteq \{v \in L^2 : p^+(x',0,\xi',D_n)v \in L^2\}.$$

In view of the fact that p is elliptic, the last set is a subset of $H^m(\mathbb{R}_+, \widetilde{E}_1)$, and we get the first part of (5.12).

According to Lemma 4.6, the space $\mathcal{E}_{(x',\xi')}$ is the dual space of $\mathcal{D}_{x',\xi')}$ with respect to the pairing induced by the L^2 inner product. This gives the second statement in (5.12).

We conclude that

$$(x',\xi') \mapsto 1 + c^*(x',\xi')\, c(x',\xi')$$

is a smooth family of operators in

$$\mathcal{L}(H^m(\mathbb{R}_+, \widetilde{E}_1), H_0^{-m}(\overline{\mathbb{R}}_+, \widetilde{E}_1)).$$

As inversion is continuous, $(1+c^*(x',\xi')c(x',\xi'))^{-1}$ also is a smooth family. Moreover, for $|\xi|\ge 1$, we have $c=c^m$, and both $c(x',\xi')$ and $c^*(x',\xi')$ are invertible. We can write

$$(1+c^*c)^{-1}=c^{-1}\,(1+(c^*)^{-1}c^{-1})^{-1}\,(c^*)^{-1}. \tag{5.14}$$

If $\varphi=\varphi(\xi')$ is an excision function on $\mathbb{R}$ which vanishes for $|\xi'|\le 1$ and is equal to one for large $|\xi'|$, then the homogeneity of c implies that, in local coordinates,

$$\varphi\, c^{-1}\in S^{-m}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};L^2(\mathbb{R}_+),H^m(\mathbb{R}_+)). \tag{5.15}$$

Similarly as in 1.6, $c^*(x',\xi')^{-1}=(c^{-1}(x',\xi'))^*$. Consequently,

$$\varphi\, c^{*-1}\in S^{-m}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};H_0^{-m}(\mathbb{R}_+),L^2(\mathbb{R}_+)). \tag{5.16}$$

Next we note that the positivity of m implies that

$$\kappa_{[\xi']}^{-1}(c^*)^{-1}(x',\xi')\,c^{-1}(x',\xi')\kappa_{[\xi']}\longrightarrow 0 \text{ in } \mathcal{L}(H_0^{-m}(\mathbb{R}_+),H^m(\mathbb{R}_+)).$$

In particular, $(1+(c^*)^{-1}c^{-1})^{-1}$ is uniformly bounded in $\mathcal{L}(L^2(\mathbb{R}_+))$. For $d=1+(c^*)^{-1}c^{-1}$ we then deduce that

$$\kappa_{[\xi']}^{-1}\left(\partial_{\xi_j}d^{-1}\right)\kappa_{[\xi']}=\kappa_{[\xi']}^{-1}d^{-1}\partial_{\xi_j}d\;d^{-1}\kappa_{[\xi']}=O([\xi']^{-1})$$

in $\mathcal{L}(L^2(\mathbb{R}_+))$. Iteration shows that

$$(1+(c^*)^{-1}c^{-1})^{-1}\in S^0(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};L^2(\mathbb{R}_+),L^2(\mathbb{R}_+)). \tag{5.17}$$

The smoothness in (x',ξ') and equations (5.14) – (5.17) then imply that

$$(1+c^*c)^{-1}\in S^{-2m}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};H_0^{-m}(\mathbb{R}_+),H^m(\mathbb{R}_+)).$$

◁

5.3. Theorem. *$(1+c^*c)^{-1}$ is a boundary symbol operator in Boutet de Monvel's calculus whose pseudodifferential part is $r^+_{x_n}(1+p^*p)|^{-1}_{x_n=0}$.*

Proof. We start with a few preliminaries. We consider the composition $c^*(x',\xi')c(x',\xi')$ with c defined in (5.11). For fixed (x,ξ), the adjoint

$$c^*(x,\xi): H_0^0(\overline{\mathbb{R}}_+)\cong L^2(\mathbb{R}_+)\to H_0^{-m}(\overline{\mathbb{R}}_+)$$

is given by

$$c^*(x',\xi')=p^*(x',0,\xi',D_n)+g^*(x',\xi')$$

with the formal adjoints of p and g.

This, however, needs some explanation. In order to keep the notation light, we will simply write $c^*=p^*(D_n)+g^*$. The adjoint g^* is of order m and class 0. It naturally maps $L^2(\mathbb{R}_+)$ to $\mathcal{S}(\mathbb{R}_+)$, which can be viewed as a subspace of $H_0^{-m}(\overline{\mathbb{R}}_+)$ via extension by zero.

In order to apply $p^*(D_n)$ to $u\in L^2(\mathbb{R}_+)$, we first extend u by zero to an element of $L^2(\mathbb{R})$. Applying $p^*(D_n)$ furnishes a distribution in $H^{-m}(\mathbb{R})$, which will in general not vanish on $\mathbb{R}_-$: It only induces a functional on $H^m(\mathbb{R}_+)$ coming from an element in $H_0^{-m}(\overline{\mathbb{R}}_+)$. Indeed, according to (1.1), we can write $p^*=s^*+q^*$ with a polynomial s^* and q^* of normal order -1. Then $q^*(D_n)$ maps $e^+L^2(\mathbb{R}_+)$ to $L^2(\mathbb{R})$ so that its restrictions to the positive and the negative half-line are defined. In view of the fact that $s^*(D_n)$ preserves the support, the distribution $p^*(D_n)u$ as an element in $H_0^{-m}(\overline{\mathbb{R}}_+)$ is given by

$$p^*(D_n)u-r^-q^*(D_n)u=p^*(D_n)u-e^-Jg^-(q^*)u$$

with the symbol $g^-(q^*)$ introduced in (3.4). It is of order m and class 0.

We therefore have

$$
\begin{aligned}
c^*c &= p^*(D_n)e^+(p^+(D_n)+g) + e^+g^*(p^+(D_n)+g)\\
&\quad -e^-Jg^-(q^*)(p^+(D_n)+g)\\
&= p^*(D_n)e^+(p^+(D_n)+g) + e^+g_1 + e^-Jg_2 \qquad (5.18)
\end{aligned}
$$

with suitable singular Green symbols g_1 and g_2 of orders $2m$ and class m. Hence (with $p = p(x',0,\xi)$)

$$
\begin{aligned}
&r^+(1+p^*p)^{-1}(D_n)\,c^*c\\
&\quad= r^+((1+p^*p)^{-1}p^*)(D_n)\,e^+p^+(D_n) + r^+((1+p^*p)^{-1}p^*)(D_n)\,e^+g\\
&\qquad + r^+(1+p^*p)^{-1}(D_n)\,e^+g_1 + r^+(1+p^*p)^{-1}(D_n)\,e^-Jg_2.
\end{aligned}
$$

Surprisingly, all the terms on the right hand side can be treated within Boutet de Monvel's calculus. The first is the composition of two truncated pseudodifferential operators. It equals

$$((1+p^*p)^{-1}p^*p)^+(D_n) - l((1+p^*p)^{-1}p^*,p),$$

where the leftover term of the composition is of order 0 and class m. The second and the third are compositions of a truncated pseudodifferential operator with a singular Green symbol, thus singular Green symbols. Both have order zero and class m. The final term is the composition of the g^+-term of the pseudodifferential part with g_2 and therefore also a singular Green symbol of order 0 and class m.

Putting all this together, we find that

$$r^+(1+p^*p)^{-1}(D_n)(1+c^*c) = 1+g_3$$

with a singular Green symbol g_3 of order 0 and class m. Hence

$$(1+c^*c)^{-1} = r^+(1+p^*p)^{-1}(D_n) - g_3(1+c^*c)^{-1}. \qquad (5.19)$$

Let us now have a look at $c^*c\,r^+((1+p^*p)^{-1})(D_n)$. We first note that we may consider e^+ a trivial action on $H_0^{-m}(\overline{\mathbb{R}}_+)$ and that

$$p^+(D_n)r^+(1+p^*p)^{-1}(D_n)e^+ \;=\; (p(1+p^*p)^{-1})^+(D_n) + l(p,(1+p^*p)^{-1}), \qquad (5.20)$$

where the leftover term on the right hand side is of order $-m$ and class 0. As the first term on the right hand side maps to $L^2(\mathbb{R})$ we can rewrite it as

$$
\begin{aligned}
&(p(1+p^*p)^{-1})(D_n)e^+ - e^-r^-(p(1+p^*p)^{-1})(D_n)e^+\\
&\quad= (p(1+p^*p)^{-1})(D_n)e^+ - e^-Jg^-(p(1+p^*p)^{-1}), \qquad (5.21)
\end{aligned}
$$

where the g^--term is of order $-m$ and class 0.

Taking into account (5.18), (5.20) and (5.21)

$$
\begin{aligned}
&(1+c^*c)r^+(1+p^*p)^{-1}(D_n)\\
&\quad= 1 + p^*(D_n)\left(e^+g_4 + e^-Jg_5\right) + e^+g_6 + e^-Jg_7
\end{aligned}
$$

with singular Green symbols g_4 and g_5 of order $-m$ and class 0 and g_6 and g_7 of order and class 0. Note that the image of the sum of the second and the fourth summand necessarily lies in $H_0^{-m}(\overline{\mathbb{R}}_+)$, since this is the case for the others.

Combining this with (5.19) we conclude that

$$\begin{aligned}
&(1+c^*c)^{-1}\\
&\quad= r^+(1+p^*p)^{-1}(D_n) - (1+c^*c)^{-1}p^*(D_n)\left(e^+g_4+e^-Jg_5\right)\\
&\qquad -(1+c^*c)^{-1}\left(e^+g_6+e^-Jg_7\right)\\
&\quad= r^+(1+p^*p)^{-1}(D_n) - r^+(1+p^*p)^{-1}(D_n)p^*(D_n)e^+g_4\\
&\qquad -r^+(1+p^*p)^{-1}(D_n)p^*(D_n)e^-Jg_5 - r^+(1+p^*p)^{-1}(D_n)\left(e^+g_6+e^-Jg_7\right)\\
&\qquad +g_3(1+c^*c)^{-1}p^*(D_n)\left(e^+g_4+e^-Jg_5\right)+g_3(1+c^*c)^{-1}\left(e^+g_6+e^-Jg_7\right).
\end{aligned}$$

The first term on the right hand side is the one we want as the pseudodifferential part. The second is the composition of a truncated pseudodifferential operator of order $-m$ with a singular Green symbol of order $-m$ and class 0, thus a singular Green symbol of order $-2m$ and class 0. The third is the composition of a g^+-type symbol of order $-m$ and class 0 with a singular Green symbol of order $-m$ and class 0, thus of the same type as the second. The summands of the fourth term are of the same type as the second and the third. As for the sum of the fifth and sixth, we note that

$$\begin{aligned}
e^+g_4+e^-Jg_5 &\in S^{-m}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{S}'(\mathbb{R}_+),\mathcal{S}(\mathbb{R}_+)\oplus\mathcal{S}(\mathbb{R}_-)) \text{ and}\\
p^*(D_n) &\in S^{m}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{S}(\mathbb{R}_+)\oplus\mathcal{S}(\mathbb{R}_-),H^{-m}(\mathbb{R})).
\end{aligned}$$

The composition of both therefore is an element of $S^0(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{S}'(\mathbb{R}_+),H^{-m}(\mathbb{R}))$. As $\mathcal{S}(\mathbb{R}_+)\oplus\mathcal{S}(\mathbb{R}_-)\hookrightarrow H^{-m}(\mathbb{R})$, we have

$$e^+g_6+e^-Jg_7\in S^0(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{S}'(\mathbb{R}_+),H^{-m}(\mathbb{R}))$$

Moreover, we know that the range of the sum of all these terms is in $H_0^{-m}(\overline{\mathbb{R}}_+)$ so that we can replace $H^{-m}(\mathbb{R})$ in the symbol space by $H_0^{-m}(\overline{\mathbb{R}}_+)$. We saw in Proposition 5.2 that

$$(1+c^*c)^{-1} \in S^{-2m}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};H_0^{-m}(\mathbb{R}_+),H^{m}(\mathbb{R}_+)).$$

As g_3 is of order 0 and class m we have

$$g_3 \in S^0(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};H^{m}(\mathbb{R}_+),\mathcal{S}(\mathbb{R}_+)).$$

Hence the total composition is an element of $S^{-2m}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{S}'(\mathbb{R}_+),\mathcal{S}(\mathbb{R}_+))$ thus a singular Green symbol of order $-2m$ and class 0. This shows that $(1+c^*c)^{-1}$ is a boundary symbol operator in Boutet de Monvel's calculus which differs from $r^+(1+p^*p)^{-1}(D_n)$ by a singular Green symbol of order $-2m$ and class 0. ◃

5.4. The inverse of $1+cc^*$**.** For fixed (x',ξ'), the operator $c(x',\xi')c^*(x',\xi'):H_0^m(\overline{\mathbb{R}}_+)\to H^{-m}(\mathbb{R}_+)$ acts on $v\in\mathcal{C}_c^\infty(\mathbb{R}_+)$ by considering the function $c^*(x',\xi')v$ as an element of $H_0^0(\overline{\mathbb{R}}_+)\cong L^2(\mathbb{R}_+)$ to which we then apply c. We know that $c^*=p^*(D_n)+g^*$ with the notation introduced above. Hence c^*v is a function in $\mathcal{S}(\mathbb{R})$; interpreting it as a distribution in $H_0^0(\overline{\mathbb{R}}_+)$ amounts to restricting it to $\mathbb{R}_+$. As e^+ can be considered a trivial action on $\mathcal{C}_c^\infty(\mathbb{R}_+)$, the action of c^* coincides with that of $p^{*,+}(D_n)+g^*$. The composition cc^* therefore coincides with the composition of two boundary symbol operators in Boutet de Monvel's calculus. As $1+cc^*$ is invertible, the inverse is also given by a boundary symbol in that calculus, and we obtain the statement below:

5.5. Corollary. *The inverse* $(1+cc^*)^{-1}$ *has the following form:*

$$(1+cc^*)^{-1} = r^+(1+pp^*)^{-1}(D_n)+g_8 \tag{5.22}$$

with a singular Green symbol g_8 *of order* $-2m$ *and class* 0*.*

5.6. Proposition. *By*

$$\mathcal{G}(p) = \begin{pmatrix} (1+p^*p)^{-1} & (1+p^*p)^{-1}p^* \\ p(1+p^*p)^{-1} & p(1+p^*p)^{-1}p^* \end{pmatrix} \in \mathcal{C}_0(T^*X, \mathcal{L}(E_1 \oplus E_2))^{\sim}$$

we denote the graph projection of p. The difference of equivalence classes

$$[\mathcal{G}(p)] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

then defines an element in $K_0(\mathcal{C}_0(T^*X, \mathcal{L}(E_1 \oplus E_2)))$ which is independent of the way the smoothing near zero is performed.

Proof. Let p_0 and p_1 be two smooth extensions of p^m. We let $p_t = (1-t)p_0 + tp_1$, $0 \le t \le 1$. For each t, p_t is a smooth function on T^*X which coincides with p^m on $\{|\xi| \ge 1\}$. The associated family $\mathcal{G}(p_t)$ is continuous in t; hence the class $[\mathcal{G}(p_t)]$ is constant. ◁

5.7. Proposition. *For the graph projection of c,*

$$\mathcal{G}(c) = \begin{pmatrix} (1+c^*c)^{-1} & (1+c^*c)^{-1}c^* \\ c(1+c^*c)^{-1} & c(1+c^*c)^{-1}c^* \end{pmatrix} \in \mathcal{C}_0(T^*\partial X, \mathcal{L}(L^2(\mathbb{R}_+, \tilde{E}_1 \oplus \tilde{E}_2)))^{\sim}$$

the difference of equivalence classes

$$[\mathcal{G}(c)] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

defines an element in $K_0(\mathcal{C}_0(T^*\partial X, \mathcal{L}(L^2(\mathbb{R}_+, \tilde{E}_1 \oplus \tilde{E}_2))))$. It does not depend on the way the smoothing near zero in 5.1 is performed.

Proof. Let p_0 and p_1 be two smooth extensions of p^m and g_0 and g_1 two smooth extensions of g^m. We let $p_t = (1-t)p_0 + tp_1$, $g_t = (1-t)g_0 + tg_1$ and

$$c_t(x', \xi') = p_t^+(x', \xi', D_n) + g_t(x', \xi'), \quad 0 \le t \le 1.$$

The associated graph projections $\mathcal{G}(c_t)$ depend continuously on t in the topology of $\mathcal{C}_0(T^*\partial X, \mathcal{L}(L^2(\mathbb{R}_+, \tilde{E}_1 \oplus \tilde{E}_2)))^{\sim}$. Hence the class is independent of t. ◁

6. The Tangent Semigroupoid

We recall a few concepts from Aastrup-Nest-Schrohe [1]

6.1. Definition. By $T^{\pm}X$ we denote the set of all vectors $(x, v) \in T\widetilde{X}|_X$ for which $\exp_x(\pm\varepsilon v) \in X$ for sufficiently small $\varepsilon > 0$. This is a semi-groupoid with addition of vectors, and $T^{\pm}X = TX^{\circ} \cup T^{\pm}X|_{\partial X}$

We define $\mathscr{T}^-X$ as the disjoint union $T^-X \cup (X \times X \times]0,1])$, endowed with the fiberwise semi-groupoid structure induced by the semi-groupoid structure on T^-X and the pair groupoid structure on $X \times X$. We glue T^-X to $X \times X \times]0,1]$ via the charts

$$T^-X \times [0,1] \supseteq U \ni (x, v, \hbar) \mapsto \begin{cases} (x, v) & \text{for } \hbar = 0 \\ (x, \exp_x(-\hbar v), \hbar) & \text{for } \hbar \neq 0 \end{cases}$$

and let $\mathscr{T}^-X(0) = T^-X$ and $\mathscr{T}^-X(\hbar) = X \times X \times \{\hbar\}$.

In order to avoid problems with the topology of $\mathscr{T}^-X$ (which is in general not a manifold with corners) we let $\mathcal{C}_c^\infty(\mathscr{T}^-X) = \mathcal{C}_c^\infty(\mathscr{T}\widetilde{X})|_{\mathscr{T}^-X}$.

C*-algebras Associated to the Semi-groupoids T^-X and $\mathscr{T}^-X$. Let $\mathcal{C}^\infty(T^-X)$ denote the smooth functions on T^-X with compact support in T^-X. We introduce

$$\begin{aligned} \pi_0 &: \mathcal{C}^\infty(T^-X) \to \mathcal{L}(L^2(TX)) \quad \text{and} \\ \pi_0^\partial &: \mathcal{C}^\infty(T^-X) \to \mathcal{L}(L^2(T^+X|_{\partial X})) \end{aligned}$$

acting by

$$\pi_0(f)\xi(x,v) = \int_{T_mX} f(x, v-w)\xi(x,w)\,dw, \tag{6.23}$$

$$\pi_0^\partial(f)\xi(x,v) = \int_{T_x^+X} f(x, v-w)\xi(x,w)\,dw. \tag{6.24}$$

Note: As f has compact support in T^-X, it naturally extends (by zero) to TX.

6.2. Definition. $C_r^*(T^-X)$ is the C^*-algebra generated by $\pi_0 \oplus \pi_0^\partial$, i.e. by the map

$$\mathcal{C}^\infty(T^-X) \ni f \mapsto (\pi_0(f), \pi_0^\partial(f)) \in \mathcal{L}(L^2(TX) \oplus L^2(T^+X|_{\partial X})).$$

6.3. Remark. According to Lemmas 2.14 and 2.15 in [1], $C_r^*(T^-X)$ has the dense $*$-subalgebra

$$\mathcal{C}_{tc}^\infty(T^-X) = \mathcal{C}_c^\infty(TX) \oplus \mathcal{C}_c^\infty(T\partial X \times \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+)$$

with the representation $\pi_0 \oplus \pi_0^\partial$ of $\mathcal{C}_c^\infty(TX)$ on $L^2(TX) \oplus L^2(T^+X|_{\partial X})$, defined as above, and the representation $\tilde\pi_0^\partial$ of $\mathcal{C}_c^\infty(T\partial X \times \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+)$ on $L^2(T^+X|_{\partial X})$ given by

$$\tilde\pi_0^\partial(K)\xi(x,v',v_n) = \int_{T^+X|_{\partial X}} K(x, v'-w', v_n, w_n)\xi(x,w',w_n)\,dw'\,dw_n.$$

6.4. An ideal in $C_r^*(T^-X)$. Denote by

$$\begin{aligned} \mathcal{F} : L^2(TX) &\to L^2(T^*X) \quad \text{and} \\ \mathcal{F}' : L^2(T\partial X) &\to L^2(T^*\partial X) \end{aligned}$$

the fiberwise Fourier transforms.

It was noted in [1, Lemma 2.15] that $\mathcal{F}' : L^2(T\partial X) \to L^2(T^*\partial X)$ provides an isomorphism between the ideal of $C_r^*(T^-X)$ generated by the representation $\tilde\pi_0^\partial$ in 6.3 and $\mathcal{C}_0(T^*\partial X, \mathcal{K})$.

The Fourier transform allows us two more important identifications: For $f \in \mathcal{C}_c^\infty(TX)$, the operator

$$\mathcal{F}\pi_0(f)\mathcal{F}^{-1} : L^2(T^*X) \to L^2(T^*X)$$

is the operator of multiplication by $\widehat{f} = \mathcal{F}f$.

At the boundary, the choice of a Riemannian metric allows us to identify $T^\pm X|_{\partial X}$ with $T\partial X \times \mathbb{R}_\pm$ and the operator

$$\mathcal{F}'\pi_0^\partial(f)\mathcal{F}'^{-1} : L^2(T^*\partial X \times \mathbb{R}_+) \to L^2(T^*\partial X \times \mathbb{R}_+)$$

with the boundary symbol operator $\mathcal{F}'(f)(x', 0, \xi', D_n)^+$.

6.5. Remark. So far we have been working with the graph projections of operators and symbols acting in vector bundles. Since we want it to give rise to elements in the K-theory of $C_r^*(T^-X)$ and $C_r^*(\mathscr{T}^-X)$ we will describe here how this can be achieved: Choose bundles F_1 and F_2 such that $E_1 \oplus F_1$ and $E_2 \oplus F_2$ are trivial with fiber $\mathbb{C}^\nu$. We can consider the graph projection of an operator A acting between sections of E_1 and E_2 as an element of $\mathcal{L}(L^2(X, \mathbb{C}^{2\nu}))$. Let π_{F_2} be the projection onto F_2. Note that

$$\mathcal{G}(A) + \pi_{F_2}$$

is a projection in $M_{2\nu}(\mathcal{K}(L^2(X))^\sim)$ and that

$$[\mathcal{G}(A) + \pi_{F_2}] - [\pi_{E_2 \oplus F_2}]$$

is the index class of A in $K_0(\mathcal{K}(L^2(X)))$. The same consideration applies to the graph projection of the symbol, so that we get a class in $K_0(C^*(T^-X))$. Since the construction is stable under the $\hbar$-scaling this gives a way of passing from general vector bundles to trivial vector bundles.

We will in the rest of the paper use this to identify various graph projections acting in vector bundles with projections in trivial bundles.

6.6. Proposition. *With the identification provided by the Fourier transforms described in 6.4, $\mathcal{G}(p) \oplus \mathcal{G}(c)$ can be regarded as an element in $M_N(C_r^*(T^-X)^\sim)$ for suitable N.*

Proof. We abbreviate $p(D_n) = p(x', 0, \xi', D_n)$ and consider the difference

$$\mathcal{G}(c) - r^+ \,\mathcal{G}(p(D_n))\, e^+ \tag{6.25}$$

between the graph projection of c and the truncated operator obtained from the graph projection of the operator $p(D_n)$, i.e. from

$$\mathcal{G}(p(D_n)) = \begin{pmatrix} (1+p^*p)^{-1}(D_n) & (1+p^*p)^{-1}(D_n)p^*(D_n) \\ p(D_n)(1+p^*p)^{-1}(D_n) & p(D_n)(1+p^*p)^{-1}(D_n)p^*(D_n) \end{pmatrix}$$

acting on $L^2(\mathbb{R}, \widetilde{E}_1 \oplus \widetilde{E}_2)$. According to Theorem 5.3, $(1+c^*c)^{-1} - r^+(1-p^*p)^{-1}(D_n)$ is a singular Green symbol of negative order and thus a compact operator from $H_0^{-m}(\overline{\mathbb{R}}_+, \widetilde{E}_1)$ to $H^m(\mathbb{R}_+, \widetilde{E}_1)$. Hence the difference (6.25) is an element of $M_N(\mathcal{C}_0(T^*\partial X, \mathcal{K}))$. As pointed out in 6.4, conjugation by the boundary Fourier transform maps it to an ideal in $M_N(C_r^*(T^-X))$. We only have to show that conjugation by the Fourier transforms maps

$$\mathcal{G}(p) \oplus r^+ \,\mathcal{G}(p(D_n))e^+$$

to an element of $M_N(C_r^*(T^-X)^\sim)$.

In order to do this, it actually suffices to find a sequence (f_k) in $\mathcal{C}_c^\infty(TX, \mathcal{L}(\widetilde{E}_1 \oplus \widetilde{E}_2))$ which converges to $f = \mathcal{F}^{-1}(\mathcal{G}(p) - e)$, where e is the usual projection onto the second component, with respect to the norm $g \mapsto \sup_{x\in X} \|g(x, \cdot)\|_{L^1}$. Indeed, this will imply that $\pi_0(f_k) \to \pi_0(f)$ or, equivalently, that as multiplication operators $\mathcal{F}f_k \to \mathcal{G}(p) - e$. Moreover, as $\|\pi_0^\partial(\cdot)\|$ is dominated by $\|\pi_0(\cdot)\|$, also the operators $r^+\mathcal{F}'(f_k)(D_n)e^+$ will approximate $r^+ \,\mathcal{G}(p(D_n))\, e^+ - e$.

It remains to find such a sequence. To this end we note that the entries in $\mathcal{G}(p) - e$ are symbols of orders $\le -m < -n$. Hence, for any $K \ge 0$, they satisfy

$$\sup_{x,v} \left|(1+|v|^2)^K \,(\mathcal{F}^{-1}q)(x,v)\right| \le \sup_{x,v} \left| \int e^{iv\xi}(1-\Delta_\xi)^K q(x,\xi)\, đ\xi \right| < \infty.$$

As $\mathcal{C}_c^\infty(TX)$ is dense in the space of functions for which the weighted sup-norm on the left hand side is finite and as this norm is larger than $\sup_x \|g(x,\cdot)\|_{L^1}$ whenever $K > N/2$, we find the desired sequence. ◁

6.7. Parametrix Construction. We fix a collar neighborhood $\partial X \times [0, 2)$ of ∂X in X. Moreover, we choose a covering of X by open sets X_j and coordinate maps $\chi_j : U_j \subset \mathbb{R}^n \to X_j$. We assume that a coordinate neighborhood either is contained in a collar neighborhood of the boundary ('boundary chart') or else does not intersect a neighborhood of the boundary ('interior chart'). It is also no restriction to suppose that the boundary

charts all lie in the collar neighborhood and that the variable normal to the boundary, x_n, is fixed and that changes of coordinates only involve the tangential variables.

We fix a partition of unity φ_j subordinate to the coordinate charts and cut-off functions ψ_j supported in the same charts with $\psi_j(x) \equiv 1$ in a neighborhood of the support of φ_j.

6.8. Lemma. *Without changing the K-classes of $\mathcal{G}(p)$ and $\mathcal{G}(c)$ we may assume that the symbol p of 5.1 is independent of x_n in a neighborhood of ∂X. In particular, we shall assume that this is the case on the collar neighborhood $\partial X \times [0,1)$ of the boundary.*

Proof. Apply a smooth deformation of the boundary variable. This will imply continuity of the change of the graph projections and thus keep the associated K-class constant. $\triangleleft$

6.9. Scaling. According to 2.14 we define

$$\begin{aligned} p_\hbar(x,\xi) &= p(x,\hbar\xi), \\ g_\hbar(x',\xi') &= \kappa_\hbar^{-1} g(x',\hbar\xi')\kappa_\hbar, \\ c_\hbar(x',\xi') &= \kappa_\hbar^{-1} \operatorname{op}^+_{x_n}(p|_{x_n=0})(x',\hbar\xi')\kappa_\hbar + g_\hbar(x',\xi') \end{aligned}$$

We denote by p^j the function p in the χ_j-coordinates and by χ_j^* the transport of operators from U_j to X_j, i.e. $\chi_j^* \operatorname{op} p^j_\hbar$ is the operator induced on X from the operator $\operatorname{op} p^j_\hbar$ on $\mathbb{R}^n$. Of course, this only makes sense when multiplied with suitable cut-off functions from the left and the right.

In the boundary charts we will use the $\hbar$-scaled boundary symbol operators $c_\hbar$; we write $c^k_\hbar$ for this symbol in the χ_k-coordinates and op' for the quantization map for boundary symbol operators.

Then we define the operator family $A_\hbar : \mathcal{C}^\infty(X, E_1) \to \mathcal{C}^\infty(X, E_2)$ by

$$A_\hbar = \sum_{\text{boundary charts}} \varphi_k \chi_k^* \operatorname{op}'(c^k_\hbar)\psi_k + \sum_{\text{interior charts}} \varphi_j \chi_j^* \operatorname{op}(p^j_\hbar)\psi_j, = A_{b,\hbar} + A_{i,\hbar}$$

consisting of a boundary part $A_{b,\hbar}$ and an interior part $A_{i,\hbar}$.

6.10. Lemma. *Let ω, ω_1 be smooth and supported in a single boundary neighborhood. Then*

$$\omega \operatorname{op}^+(p_\hbar)\omega_1 = \omega\kappa_\hbar^{-1} \operatorname{op}'((\operatorname{op}^+_{x_n} p|_{x_n=0})(x',\hbar\xi'))\kappa_\hbar\omega_1.$$

Proof. This follows from the fact that p is independent of x_n on the collar neighborhood and the computation in 3.1. $\triangleleft$

6.11. Lemma. *$A_\hbar$ is an operator family in Boutet de Monvel's calculus. Its pseudodifferential symbol is of the form $p_\hbar + \hbar q(\hbar)$, where $q_{1/\hbar}$ is bounded in S^{m-1}_{tr}. Over a boundary chart, its singular Green symbol is of the form $g_\hbar + \hbar r(\hbar)$ with $r_{1/\hbar}$ bounded in $S^{m-1}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{S}'(\mathbb{R}_+), \mathcal{S}(\mathbb{R}_+))$.*

Proof. This follows from 2.13, in particular (2.16), in connection with Lemma 6.10. $\triangleleft$

We shall say that a family $C(\hbar)$ of operators given by pseudodifferential operators in the interior and operator-valued symbols close to the boundary is semiclassically bounded of order μ, if, for the family $q(\hbar)$ of pseudodifferential symbols, $q_{1/\hbar}$ is bounded in S^μ, and, for the family $d(\hbar)$ of operator-valued symbols, $d_{1/\hbar}$ is bounded in $S^\mu(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; E, F)$, where E and F have to be specified.

6.12. Proposition. *We have*

$$\begin{aligned}
1 + A_\hbar^* A_\hbar = & \sum_{\textit{boundary charts}} \varphi_k \chi_k^* \operatorname{op}'(1 + c_\hbar^{k*} c_\hbar^k)\psi_k \\
& + \sum_{\textit{interior charts}} \varphi_j \chi_j^* \operatorname{op}(1 + p_\hbar^{j*} p_\hbar^j)\psi_j + \hbar R_1(\hbar) \\
= & \sum_{\textit{boundary charts}} \varphi_k \chi_k^* \operatorname{op}'(1 + c^{k*} c^k)_\hbar \psi_k \\
& + \sum_{\textit{interior charts}} \varphi_j \chi_j^* \operatorname{op}(1 + p^{j*} p^j)_\hbar \psi_j + \hbar R_1(\hbar),
\end{aligned}$$

where the family R_1 is semiclassically bounded of order $2m - 1$ with operator-valued symbols acting between $E = H^m(\mathbb{R}_+)$ and $F = H_0^{-m}(\overline{\mathbb{R}}_+)$.
Here, $p^{j,*}$ is the adjoint of the symbol p^j in local coordinates, and $c^{k,*}$ is the adjoint of the operator-valued symbol c^k.

Proof. For the first identity apply 2.14(c) using Lemma 6.11. The second is obvious. $\triangleleft$

6.13. Proposition. *Let $\varphi, \psi \in \mathcal{C}_c^\infty(\partial X \times (0,1))$ be supported in the intersection of an interior and a boundary neighborhood. Then*

$$\varphi \operatorname{op}'(1 + c^* c)_\hbar^{-1} \psi - \varphi \operatorname{op}(1 + p^* p)_\hbar^{-1} \psi$$

is a semiclassically regularizing pseudodifferential operator; i.e. for each N, we can write it as $\hbar^N \operatorname{op} r^N(\hbar)$ with r^N bounded in S^{-2m-N}.

Proof. We know that $(1 + c^* c)^{-1}$ is a boundary symbol in Boutet de Monvel's calculus, whose pseudodifferential part is given by $(1 + p|_{x_n=0}^* p|_{x_n=0})^{-1}$. As the localization of the $\hbar$-scaled singular Green part to the interior is semiclassically smoothing by Lemma 3.3, this implies the assertion. $\triangleleft$

6.14. Proposition. *Define*

$$B(\hbar) = \sum_{\textit{boundary charts}} \varphi_k \chi_k^* \operatorname{op}'(1 + c^{k*} c^k)_\hbar^{-1} \psi_k + \sum_{\textit{interior charts}} \varphi_j \chi_j^* \operatorname{op}(1 + p^{j*} p^j)_\hbar^{-1} \psi_j \tag{6.26}$$

This is an operator family in Boutet de Monvel's calculus which is semiclassically bounded of order $-2m$, and

$$\begin{aligned}
B(\hbar)(1 + A_\hbar^* A_\hbar) &= 1 + \hbar R_2(\hbar) \quad \textit{and} \\
(1 + A_\hbar^* A_\hbar) B(\hbar) &= 1 + \hbar R_3(\hbar)
\end{aligned}$$

with families R_2, R_3 which are semiclassically bounded of order -1 with operator-valued symbols acting on $H^m(\mathbb{R}_+)$ for R_2 and on $H_0^{-m}(\overline{\mathbb{R}}_+)$ for R_3.

Proof. This follows from 2.14(c) in connection with Propositions 6.12 and 6.13. $\triangleleft$

6.15. Corollary. *We infer from Proposition 6.14 that*

$$(1 + A_\hbar^* A_\hbar)^{-1} = B(\hbar) - \hbar B(\hbar) R_3(\hbar) + \hbar^2 R_2(\hbar)(1 + A_\hbar^* A_\hbar)^{-1} R_3(\hbar). \tag{6.27}$$

From this we want to deduce that $(1+A_\hbar^*A_\hbar)^{-1}$ differs from $B(\hbar)$ by a term which is $O(\hbar)$ in $\mathcal{L}(L^2(X))$. So far, this is not obvious: The boundary symbol parts of R_2 and R_3 act on H^m and H_0^{-m}, respectively, while we only can guarantee boundedness of the inverse on $L^2(X)$. To this end we make the following observations:

6.16. Lemma. *Given $N \in \mathbb{N}$ we find an operator family $C(\hbar)$, $0 < \hbar \le 1$, in Boutet de Monvel's calculus such that*

$$C(\hbar)A_\hbar = 1 + S_N(\hbar)$$

with C and S_N semiclassically bounded of orders $-m$ and $-N$, respectively, in Boutet de Monvel's calculus.

Proof. Apply a semiclassical parametrix construction in Boutet de Monvel's calculus, using Proposition 3.8. ◁

6.17. Lemma. *The operator families $(1 + A_\hbar^*A_\hbar)^{-1}$, $(1 + A_\hbar A_\hbar^*)^{-1}$, $A_\hbar(1 + A_\hbar^*A_\hbar)^{-1}A_\hbar^*$, $A_\hbar(1+A_\hbar^*A_\hbar)^{-1}$, and $(1+A_\hbar^*A_\hbar)^{-1}A_\hbar^*$ are (after continuous extension) uniformly bounded on the corresponding L^2 spaces.*

Proof. For the first two families the statement is obvious as their operator norm is bounded by 1. For the third we use that, by Lemma 4.13,

$$A_\hbar(1 + A_\hbar^*A_\hbar)^{-1}A_\hbar^* = 1 - (1 + A_\hbar A_\hbar^*)^{-1}.$$

For the fourth we note that on a Hilbert space, the norm of an operator T equals $\|T^*T\|^{1/2}$. We apply this to $T = A_\hbar(1 + A_\hbar^*A_\hbar)^{-1}$. By 4.8,

$$T^*T = (1 + A_\hbar^*A_\hbar)^{-1}A_\hbar^*A_\hbar(1 + A_\hbar^*A_\hbar)^{-1} = (1 + A_\hbar^*A_\hbar)^{-1} - (1 + A_\hbar^*A_\hbar)^{-2},$$

which is bounded. Duality yields the boundedness of the fifth family. ◁

6.18. Corollary. $(1 + A_\hbar^*A_\hbar)^{-1} - B(\hbar) = O(\hbar)$ *in* $\mathcal{L}(L^2(X, E_1))$.

Proof. In the third term on the right hand side of (6.27) we can write

$$(1 + A_\hbar^*A_\hbar)^{-1} = C(\hbar)A_\hbar(1 + A_\hbar^*A_\hbar)^{-1}A_\hbar^*C(\hbar)^* \\ -S_N(\hbar)(1 + A_\hbar^*A_\hbar)^{-1}A_\hbar^*C(\hbar)^* - C(\hbar)A_\hbar(1 + A_\hbar^*A_\hbar)^{-1}S_N(\hbar)^* \tag{6.28}$$

for some $N > m$. By Lemma 2.17, R_2C, R_2S_N C^*R_3 and $S_N^*R_3$ are bounded on L^2, uniformly in $\hbar$. The assertion then follows from (6.27). ◁

6.19. Corollary. $A_\hbar(1 + A_\hbar^*A_\hbar)^{-1} - A_\hbar B(\hbar) = O(\hbar)$ *in* $\mathcal{L}(L^2(X, E_1), L^2(X, E_2))$.

Proof. We multiply equation 6.27 from the left by $A_\hbar$. The composition $A_\hbar B(\hbar)$ furnishes an operator family in Boutet de Monvel's calculus which is semiclassically bounded of order $-m$. According to Lemma 2.17, the operator family $A_\hbar B(\hbar)R_3(\hbar)$ is therefore uniformly bounded on L^2. In the second term on the right hand side we substitute according to equation (6.28). We note that $A_\hbar R_2C$ and $A_\hbar R_2S_N$ are semiclassically bounded of order 0, with the operator-valued symbols acting on $L^2(\mathbb{R}_+)$. Hence $A_\hbar R_2(\hbar)(1+A_\hbar^*A_\hbar)^{-1}R_3(\hbar)$ is uniformly bounded in $\mathcal{L}(L^2)$. ◁

In an analogous way we find:

6.20. Corollary. $(1 + A_\hbar^*A_\hbar)^{-1}A_\hbar^* - B(\hbar)A_\hbar^* = O(\hbar)$ *in* $\mathcal{L}(L^2(X, E_2), L^2(X, E_1))$.

6.21. Remark. We know from Lemma 4.13 that, as operators on $L^2(X, E_2)$,

$$A_\hbar(1 + A_\hbar^*A_\hbar)^{-1}A_\hbar^* = 1 - (1 + A_\hbar A_\hbar^*)^{-1}.$$

In order to determine the structure of the left hand side, we construct, similarly as before, an approximate inverse $\widetilde{B}(\hbar)$ to $1+A_\hbar A_\hbar^*$, using the structure of the boundary symbol operator $(1+cc^*)^{-1}$ determined in Corollary 5.5. We obtain the following result:

6.22. Proposition.

$$\widetilde{B}(\hbar) = \sum_{\textit{boundary charts}} \varphi_k \chi_k^* \operatorname{op}'(1+c^k c^{k*})_\hbar^{-1}\psi_k + \sum_{\textit{interior charts}} \varphi_j \chi_j^* \operatorname{op}(1+p^j p^{j*})_\hbar^{-1}\psi_j \tag{6.29}$$

defines a semiclassically bounded family in Boutet de Monvel's calculus of order $-2m$*, and*

$$(1+A_\hbar A_\hbar^*)^{-1} = \widetilde{B}(\hbar) + \hbar R_4(\hbar);$$

with a family R_4 *which is uniformly bounded on* $L^2(X,E_1)$*.*

6.23. Definition. The construction of $A_\hbar$ together with Propositions 6.6 allows us to define a section of the continuous field $M_N(C_r^*(\mathscr{T}^-X)^\sim)$ by

$$s(\hbar) = \begin{cases} \mathcal{G}(A_\hbar), & 0<\hbar\le 1 \\ \mathcal{G}(p)\oplus\mathcal{G}(c), & \hbar = 0 \end{cases}.$$

We shall now show that this section is continuous. We will distinguish the cases $\hbar>0$ and $\hbar=0$.

6.24. Proposition. *The section* s *is continuous on* $(0,1]$*.*

Proof. It follows from 2.14(a) and the fact that the symbol topology is stronger than the operator topology that the mappings

$$\begin{aligned} \hbar \mapsto A_\hbar &\in \mathcal{L}(H^m(X,E_1), L^2(X,E_2)) \quad \text{and} \\ \hbar \mapsto A_\hbar &\in \mathcal{L}(L^2(X,E_1), H_0^{-m}(X,E_2)) \end{aligned}$$

depend continuously on $\hbar$. As taking adjoints and inversion are continuous, we obtain the assertion. ◁

6.25. Theorem. *The section* s *is continuous in* $\hbar=0$*.*

Proof. Consider the four entries of $\mathcal{G}(A_\hbar)$. By Corollary 6.15, $(1+A_\hbar^*A_\hbar)^{-1}$ differs from $B(\hbar)$ by a term which vanishes in $\hbar=0$. It is therefore enough to show the continuity of $B(\hbar)$. By construction, the boundary symbol of $B(\hbar)$ is given by $(1+c^*c)_\hbar^{-1}$ while the interior symbol is $(1+p^*p)_\hbar^{-1}$.

By Proposition 2.18 we find smoothing symbols q_k converging to $(1+p^*p)^{-1}$ in the topology of S^0. Since we assumed p to be constant near ∂X, we may assume the same of the q_k.

As for the boundary symbol, we know that $(1+c^*c)^{-1}$ is a boundary symbol operator in Boutet de Monvel's calculus whose pseudodifferential part is $r^+(1+p^*p)|_{x_n=0}^{-1}(D_n)$. Denote by h its singular Green part. According to Proposition 2.18 we find a sequence of symbols h_k in $S^{-\infty}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{S}'(\mathbb{R}_+),\mathcal{S}(\mathbb{R}_+))$ converging to h in the topology of $S^0(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};L^2(\mathbb{R}_+),L^2(\mathbb{R}_+))$. Replacing, for $\hbar=1$, in the definition of $B(1)$ the pseudodifferential symbols over interior charts by q_k and the boundary symbol operators over boundary charts by $r^+q_k|_{x_n=0}(D_n)+h_k$, we obtain a sequence of operators B_k. According to Remark 2.20, a further approximation allows us to assume that B_k are integral operators with smooth compactly supported integral kernels. Similarly we approximate

the other entries in $\mathcal{G}(A)-e$. Adding then e again and going over to $\hbar$-scaled symbols, we infer from Lemma 2.17 that the approximation is uniform for $0<\hbar\le 1$. Hence we obtain a sequence of sections $s_k(\hbar)$ of $M_N(C_r^*(\mathscr{T}X))$ which approximates $\mathcal{G}(A_\hbar)$ uniformly.

By definition, these sections are continuous for $\hbar>0$ and have a continuous extension to $\hbar=0$ given by the $N\times N$-matrices of their interior and boundary symbols. As this matrix, on the other hand, tends to $\mathcal{G}(p)\oplus\mathcal{G}(c)$, we conclude that s is continuous. ◁

7. The Fundamental Class for Manifolds with Boundary

In this section we will describe how the fundamental class

$$\int_{T^*X^\circ} : H_c^*(T^*X^\circ) = HP^*(\mathcal{C}_c^\infty(T^*X^\circ) \to \mathbb{C}$$

extends to a fundamental class

$$F : HP^*(\mathcal{C}_{tc}^\infty(T^-X)) \to \mathbb{C}.$$

Let us first assume $T^*X = T^*\partial X \times T^*\mathbb{R}_+$ so that we can consider the elements of $\mathcal{C}_c^\infty(T^-X)$ as elements of $\mathcal{C}_c^\infty(T^*\partial X)\hat{\otimes}_\pi \mathcal{C}_{tc}^\infty(T^-\mathbb{R}_+)$. We write an element of $\mathcal{C}_{tc}^\infty(T^-\mathbb{R}_+)$ as the sum of a pseudodifferential symbol p and a singular Green symbol g on the boundary. We then obtain the operator of multiplication by p on $T^*\mathbb{R}_+$ and the boundary symbol operator

$$c = p(0,D_n) + g. \tag{7.30}$$

Following Fedosov we define

$$\operatorname{tr}'(p+g) = \operatorname{tr}(g),$$

noting that g is an integral operator on $L^2(\mathbb{R}_+)$ with a rapidly decreasing kernel and thus trace class. The functional tr' is not quite a trace, but satisfies the following fundamental property [5, (2.19)]

$$\operatorname{tr}'([p_1+g_1, p_2+g_2]) \;=\; -i\int \frac{\partial p_1(0,\xi_n)}{\partial \xi_n} p_2(0,\xi_n)d\xi_n = i\int p_1(0,\xi_n)\frac{\partial p_2(0,\xi_n)}{\partial \xi_n} d\xi_n.$$

Given an element in the cyclic periodic complex, i.e. an element in $\mathcal{C}_{tc}^\infty(T^-X)^{\otimes^{m+1}}$, we first introduce the boundary functional

$$\begin{aligned} &F_\partial((f_0\otimes(p_0+g_0))\otimes(f_1\otimes(p_1+g_1))\otimes\cdots\otimes(f_m\otimes(p_m+g_m)))\\ &= \int_{T^*\partial X} f_0\, df_1\cdots df_m \operatorname{tr}'((p_0+g_0)\cdots(p_m+g_m)).\end{aligned}$$

We have used here the splitting of $\mathcal{C}_{tc}^\infty(T^-X)$ into tensor factors. For notational convenience we will omit the tensor symbols and write $f_j(p_j+g_j)$ instead of $f_j\otimes(p_j+g_j)$.

The fundamental class is given by

$$\begin{aligned} F(f_0(p_0+g_0)\otimes f_1(p_1+g_1)\otimes\cdots\otimes f_m(p_m+g_m)) &= \int_{T^*X} f_0p_0 d(f_1p_1)\cdots d(f_mp_m)\\ +iF_\partial\Big(\sum_{\sigma \text{ cyclic}} \operatorname{sgn}(\sigma) f_{\sigma(0)}(p_{\sigma(0)}+g_{\sigma(0)})\otimes\cdots\otimes f_{\sigma(m)}&(p_{\sigma(m)}+g_{\sigma(m)})\Big).\end{aligned}$$

7.1. Proposition. *The fundamental class F is a cocycle on the periodic cyclic complex.*

Proof. We need to prove that $F((B+b)\underline{a})=0$, where $\underline{a}\in CC_*^{per}(\mathcal{C}_{tc}^\infty(T^-X))$. By Stokes' theorem the boundary part F_∂ of F vanishes on $B\underline{a}$. The remaining 'nonboundary' part of F clearly vanishes on $b\underline{a}$. Computing the nonboundary part we get

$$\int_{T^*X} d(f_0p_0)d(f_1p_1)\cdots d(f_mp_m)=\int_{\partial(T^*X)} f_0p_0\,d(f_1p_1)\cdots d(f_mp_m)$$

We want to compute the boundary part, i.e. F_∂, on the cyclic permuted terms appearing in F. A single cyclic permutation of $b(\underline{a})$ without $\operatorname{sgn}(\sigma)$ is of the form

$$\begin{aligned}
&f_{i+1}(p_{i+1}+g_{i+1})\otimes\cdots\otimes f_0(p_0+g_0)f_1(p_1+g_1)\otimes\cdots\otimes f_i(p_i+g_i)\\
&-f_{i+1}(p_{i+1}+g_{i+1})\otimes\cdots\otimes f_1(p_1+g_1)f_2(p_2+g_2)\otimes\cdots\otimes f_i(p_i+g_i)\\
&\qquad\qquad\vdots\\
&+(-1)^{i-1}f_{i+1}(p_{i+1}+g_{i+1})\otimes\cdots\otimes f_0(p_0+g_0)\otimes\cdots\otimes f_{i-1}(p_{i-1}+g_{i-1})f_i(p_i+g_i)\\
&+(-1)^{i}f_i(p_i+g_i)f_{i+1}(p_{i+1}+g_{i+1})\otimes\cdots\otimes f_0(p_0+g_0)\otimes\cdots\otimes f_{i-1}(p_{i-1}+g_{i-1})\\
&\qquad\qquad\vdots\\
&+(-1)^{m}f_i(p_i+g_i)\otimes\cdots\otimes f_m(p_m+g_m)f_0(p_0+g_0)\otimes\cdots\otimes f_{i-1}(p_{i-1}+g_{i-1}).
\end{aligned}$$

We split this expression into the sum of the first i terms and the sum of the subsequent $m+1-i$ terms. The action of F_∂ on the first i terms is

$$\begin{aligned}
&\operatorname{tr}'\big((p_{i+1}+g_{i+1})(p_{i+2}+g_{i+2})\cdots(p_m+g_m)(p_0+g_0)\cdots(p_i+g_i)\big)\\
&\quad\times\int_{T^*\partial X}\Big(f_{i+1}df_{i+2}\cdots d(f_0f_1)\cdots df_i-f_{i+1}df_{i+2}\cdots d(f_1f_2)\cdots df_i+\ldots\\
&\qquad+\ldots(-1)^{i-1}f_{i+1}df_{i+2}\cdots d(f_{i-1}f_i)\Big).
\end{aligned}$$

The second factor in this expression can be rewritten as

$$\begin{aligned}
&\int_{T^*\partial X}\Big(f_0f_{i+1}df_{i+2}\cdots df_mdf_1\cdots df_i+f_1f_{i+1}df_{i+2}\cdots df_0df_2\cdots df_i\\
&\qquad-\big(f_1f_{i+1}df_{i+2}\cdots df_0df_2\cdots df_i+f_2f_{i+1}df_{i+2}\cdots df_1df_3\cdots df_i\big)\\
&\qquad+\ldots+(-1)^{i-1}\big(f_{i-1}f_{i+1}df_{i+2}\cdots df_{i-2}df_i+f_if_{i+1}df_{i+2}\cdots df_{i-1}\big)\Big)
\end{aligned}$$

$$=\int_{T^*\partial X}\Big(f_0f_{i+1}df_{i+2}\cdots df_mdf_1\cdots df_i+(-1)^{i-1}f_if_{i+1}df_{i+2}\cdots df_{i-1}\Big). \tag{7.31}$$

A short computation shows that

$$\begin{aligned}
d(f_0f_if_{i+1})df_{i+2}\cdots df_mdf_1\cdots df_{i-1}&=f_0f_idf_{i+1}\cdots df_mdf_1\cdots df_{i-1}\\
&\quad+(-1)^mf_0f_{i+1}df_{i+2}\cdots df_mdf_1\cdots df_i+(-1)^{m-(i-1)}f_if_{i+1}df_{i+2}\cdots df_{i-1}.
\end{aligned}$$

Stokes' theorem then implies that the term (7.31) equals

$$\begin{aligned}
&(-1)^{m+1}\int_{T^*\partial X}f_0f_idf_{i+1}\cdots df_mdf_1\cdots df_{i-1}\\
&\quad=\ (-1)^{mi+1}\int_{T^*\partial X}f_0f_idf_1\cdots df_{i-1}df_{i+1}\cdots df_m.
\end{aligned}$$

Putting everything together, the action of F_∂ on the first sum is given by

$$\begin{aligned}
(-1)^{mi+1}\quad&\operatorname{tr}'\big((p_{i+1}+g_{i+1})(p_{i+2}+g_{i+2})\cdots(p_m+g_m)(p_0+g_0)\cdots(p_i+g_i)\big)\\
&\times\int_{T^*\partial X}f_0f_idf_1\cdots df_{i-1}df_{i+1}\cdots df_m.
\end{aligned}$$

The action of F_∂ on the remaining $m+1-i$ terms gives

$$\begin{aligned}
&\operatorname{tr}'\big((p_i+g_i)(p_{i+1}+g_{i+1})\cdots(p_m+g_m)(p_0+g_0)\cdots(p_{i-1}+g_{i-1})\big)\\
&\qquad\times\int_{T^*\partial X}\Big((-1)^i f_i f_{i+1} df_{i+2}\cdots df_m df_0\cdots df_{i-1}+\\
&\qquad\quad(-1)^{i+1}\big(f_i f_{i+1} df_{i+2}\cdots df_{i-1}+f_i f_{i+2} df_{i+1} df_{i+3}\cdots df_{i-1}\big)+\ldots\\
&\qquad\quad+(-1)^m\big(f_i f_m df_{i+2}\cdots df_{m-1} df_0\cdots df_{i-1}+f_i f_0 df_{i+2}\cdots df_m df_1\cdots df_{i-1}\big)\\
&= (-1)^{mi}\operatorname{tr}'\Big((p_i+g_i)(p_{i+1}+g_{i+1})\cdots(p_m+g_m)(p_0+g_0)\cdots(p_{i-1}+g_{i-1})\Big)\\
&\qquad\times\int_{T^*\partial X} f_i f_0 df_1\cdots df_{i-1} df_{i+1}\cdots df_m.
\end{aligned}$$

All in all we get that the action of F_∂ on this symmetrization is

$$\begin{aligned}
&(-1)^{mi}\operatorname{tr}'([p_i+g_i,(p_{i+1}+g_{i+1})\cdots(p_m+g_m)(p_0+g_0)\cdots(p_{i-1}+g_{i-1})])\\
&\times\int_{T^*\partial X} f_i f_0 df_1\cdots df_{i-1} df_{i+1}\cdots df_m\\
&= (-1)^{mi+1}\, i\int_{T^*\partial X}\int_{\mathbb{R}} p_0(0,\xi_n)\cdots p_{i-1}(0,\xi_n)p_{i+1}(0,\xi_n)\cdots p_m(0,\xi_n)\ \frac{\partial p_i(0,\xi_n)}{\partial\xi_n}d\xi_n\\
&\qquad\times f_i f_0 df_1\cdots df_{i-1} df_{i+1}\cdots df_m\\
&= (-1)^{(m+1)i}\, i\int_{\partial(T^*X)} p_0(0,\xi_n)\cdots p_{i-1}(0,\xi_n)p_{i+1}(0,\xi_n)\cdots p_m(0,\xi_n)\ f_i f_0 df_1\cdots df_{i-1}\\
&\qquad\times\frac{\partial p_i(0,\xi_n)}{\partial\xi_n}d\xi_n df_{i+1}\cdots df_m\\
&= \operatorname{sgn}(\sigma)\, i\int_{\partial(T^*X)} p_0(0,\xi_n)\cdots p_{i-1}(0,\xi_n)p_{i+1}(0,\xi_n)\cdots p_m(0,\xi_n)\ f_i f_0 df_1\cdots df_{i-1}\\
&\qquad\times\frac{\partial p_i(0,\xi_n)}{\partial\xi_n}d\xi_n df_{i+1}\cdots df_m,
\end{aligned}$$

where σ is the corresponding permutation. Hence

$$\begin{aligned}
F(b(\underline{a})) &= \sum_{i=1}^m\int_{\partial(T^*X)} p_0(0,\xi_n)\cdots p_{i-1}(0,\xi_n)p_{i+1}(0,\xi_n)\cdots p_m(0,\xi_n)\ f_i f_0 df_1\cdots df_{i-1}\\
&\qquad\times\frac{\partial p_i(0,\xi_n)}{\partial\xi_n}d\xi_n df_{i+1}\cdots df_m,
\end{aligned}$$

which is equal to

$$\int_{\partial(T^*X)} f_0 p_0 d(f_1p_1)\cdots d(f_mp_m).$$

◁

7.2. The fundamental class in general. For general X the restriction of an element a in $\mathcal{C}^\infty_{tc}(T^-X)$ to $\partial(T^*X)$ can be factorized as a sum of elements of the form $f\otimes(p+g)$, i.e. the boundary symbol factorizes. We will adopt this notation, i.e. the boundary part of a will be denoted $f\otimes(p+g)$. The symbol part in the interior will be denoted by $\tilde{a}$, which is a function on T^*X. It is then straightforward to generalize the fundamental class to nonproduct cases:

Let ω be a closed differential form on T^*X of even degree, which is the pull back of a closed differential form on X. We first define $F_{\partial,\omega}$ by

$$\begin{aligned} &F_{\partial,\omega}(f_0 \otimes (p_0+g_0)) \otimes (f_1 \otimes (p_1+g_1) \otimes \cdots \otimes (f_m \otimes (p_m+g_m))) \\ &= \int_{T^*\partial X} f_0 df_1 \cdots df_m \cdot \omega \cdot \operatorname{tr}'((p_0+g_0)\cdots(p_m+g_m)) \end{aligned}$$

and then let

$$\begin{aligned} &F(a_0 \otimes a_1 \otimes \cdots \otimes a_m) \\ &= \int_{T^*X} \tilde{a}_0 d\tilde{a}_1 \cdots \tilde{a}_m \cdot \omega - iF_{\partial,\omega}\left(\sum_{\sigma \text{ cyclic}} \operatorname{sgn}(\sigma) a_0 \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(m)} \right). \end{aligned}$$

7.3. Proposition. *F_ω descends to a map*

$$F_\omega : HP(\mathcal{C}^\infty_{tc}(T^-X)) \to \mathbb{C}.$$

Proof. The same computation as in Proposition 7.1. ◁

The Index Formula

Like in [3, Section 2.5] the short exact sequence

$$0 \to \mathcal{C}_0(]0,1]) \otimes \mathcal{K} \to C^*_r(\mathscr{T}^-X) \to C^*_r(T^-X) \to 0$$

induces an analytic index map

$$\operatorname{ind}_a : K_0(C^*_r(T^-X)) \to \mathbb{Z}.$$

In this section we will give a formula for this index map.

We have another short exact sequence coming from the interior of the manifold, namely

$$0 \to \mathcal{C}_0(]0,1]) \otimes \mathcal{K} \to C^*_r(\mathscr{T}X^\circ) \to \mathcal{C}_0(T^*X^\circ) \to 0$$

(noting that $C^*_r(TX^\circ) \cong \mathcal{C}_0(T^*X^\circ)$) also inducing an analytic index map

$$\operatorname{ind}_a : K_0(\mathcal{C}_0(T^*X^\circ)) \to \mathbb{Z}.$$

According to Connes [3, Section 2.5] we have in this case

$$\operatorname{ind}_a = \operatorname{ind}_t,$$

where ind_t denotes the topological index. On the other hand we have an isomorphism $\Phi : K_0(C^*_r(T^-X)) \to K_0(\mathcal{C}_0(T^*X^\circ))$ and the diagram

$$\begin{array}{ccc} K_0(C^*_r(T^-X)) & \to & \mathbb{Z} \\ \downarrow & \nearrow & \\ K_0(\mathcal{C}_0(T^*X^\circ)) & & \end{array}$$

commutes.

Let T be the Fredholm operator of order and class zero in Boutet de Monvel's calculus introduced in 4.10, and let $\mathcal{G}(a) = \mathcal{G}(p) \oplus \mathcal{G}(c)$ denote the graph projection of a complete symbol of $\Lambda^{m,+}_- T$. From Theorem 6.25 we obtain

7.4. Theorem. *The index of T is given by*

$$\operatorname{ind} T = \operatorname{ind}_a([\mathcal{G}(a)] - [e]).$$

We define the topological index

$$\mathrm{ind}_t : K_0(C_r^*(T^-X)) \to \mathbb{Z}$$

as the composition $\mathrm{ind}_t \circ \Phi$. We thus get an index theorem

7.5. Theorem. $\mathrm{ind}_a = \mathrm{ind}_t$.

With this notation we can now prove

7.6. Theorem.

$$\mathrm{ind}\, T = F_{Td(X)}(\mathrm{ch}([\mathcal{G}(a)] - [e])),$$

where ch *denotes the the Chern-Connes character.*

Proof. Let $C^n(T^-X)$ denote the subalgebra of $C_r^*(T^-X)$ consisting of symbols of order strictly less than $-n$, n being the dimension of X. We note that $C^n(T^-X)^\sim$ is closed under holomorphic functional calculus and hence $K_0(C^n(T^-X)) = K_0(C_r^*(T^-X))$. Also note that F_ω is defined on $HP(C^n(T^-X))$. Using the cohomological form of the topological index we get the following commutative diagrams:

$$\begin{tikzcd}
K_0(C^n(T^*X^\circ)) \arrow[r, equal] \arrow[d, "\mathrm{ch}"] & K_0(C^nT^-(X)) \arrow[d, "\mathrm{ch}"] \\
HP_{ev}(C^n(T^*X^\circ)) \arrow[r] \arrow[d, equal] & HP_{ev}(C^n(T^-X))\cdot \arrow[d, "F_{Td(X)}"] \\
H_c^*(T^*X^\circ) \arrow[r, "\int Td(X)\cdot"] & \mathbb{Z}
\end{tikzcd}$$

and

$$\begin{tikzcd}
K_0(C^n(T^*X^\circ)) \arrow[r, equal] \arrow[d, "\mathrm{ch}"'] & K_0(\mathcal{C}_0(T^*X^\circ)) \arrow[r, equal] \arrow[drr, "\mathrm{ind}_t"] & K_0(\mathcal{C}_0(T^*X^\circ)) \arrow[dr, "\mathrm{ind}_a"] & \\
HP_{ev}(C^n(T^*X^\circ)) \arrow[r, equal] & H_c^*(T^*X^\circ) \arrow[rr, "\int Td(X)\cdot"'] & & \mathbb{Z}
\end{tikzcd}$$

and

$$\begin{tikzcd}
K_0(\mathcal{C}_0(T^*X^\circ)) \arrow[r, equal] \arrow[dr, "\mathrm{ind}_a"'] & K_0(C_r^*(T^-X)) \arrow[r, equal] \arrow[d, "\mathrm{ind}_a"] & K_0(C^n(T^-X)) \arrow[dl, "\mathrm{ind}_a"] \\
& \mathbb{Z} &
\end{tikzcd}$$

from which follows that $F_{Td(X)} \circ \mathrm{ch} = \mathrm{ind}_a$ on $K_0(C^n(T^-X))$. ◁

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